A Selected proofs

A.1 Proof of Proposition 1

A.1.1 The planner's problem

The planner's objective is

$$W(\varphi, \gamma \,|\, N) = \int U\left[g(x \,|\, \varphi, \gamma)\right] \, dN(x).$$

where we ignore participation cost to simplify notations (because they are sunk), and where we make it explicit that post-trade exposures depend on centralized and OTC market trades (φ, γ) via (6). Given that the measure N is finite, $N(X) < \infty$, repeat applications of the Cauchy Schwartz inequality show that the planner's objective is continuous in (γ, φ). Because $(\gamma, \varphi) \mapsto W(\gamma, \varphi \,|\, N)$ is continuous, it is lower semi-continuous. Clearly, the function is also concave and the constraint set is closed and bounded. Existence of a solution then follows from an application of Proposition 1.2, Chapter II in Eckland and Témam (1987).

The uniqueness of post-trade exposures follows because the objective is strictly concave in post-trade exposures.

The derivation of the first-order condition is based on arguments from Chapter 8 in Luenberger (1969). Following the notations of Luenberger, we let Ω denote the set of square-integrable trades (γ, φ) that are feasible N-almost everywhere, and we view (4) as a parametric linear equality constraint. The separating-hyperplane argument of Theorem 1 in Chapter 8.3, together with its generalization to linear equality constraint in Problem 7, imply that:

Lemma 7. *There exists some $g_c \in \mathbb{R}$ such that the solution to the planner's problem maximizes the Lagrangian $L(\varphi, \gamma) \equiv W(\varphi, \gamma) - U_g(g_c) \int \varphi(x) \, dN(x)$.*

We provide a detailed proof in online Appendix B.3. We now derive first-order condition by applying a variational argument to the Lagrangian. We start with first-order conditions with respect to centralized market trades. Given any optimal (φ, γ) and we let:

$$\begin{aligned}
\hat{\varphi}(x) &= \varphi(x) + \varepsilon \left[\Phi(x) - \varphi(x)\right] \mathbb{I}_{\{g(x) < g_c\}} - \varepsilon \left[\Phi(x) + \varphi(x)\right] \mathbb{I}_{\{g(x) > g_c\}} \\
&\equiv \varphi(x) + \varepsilon \Delta_{\varphi}(x),
\end{aligned}$$

and we note that $(\hat{\varphi}, \gamma)$ is feasible for the planning problem, as long as $\varepsilon \in [0,1]$. Hence, for small ε, we obtain that up to second-order terms:

$$
\begin{aligned}
&L(\hat{\varphi},\gamma) - L(\varphi,\gamma) = \varepsilon \int U_g\left[g(x)\right]\Delta_\varphi(x)\,dN(x) - \varepsilon U_g\left(g_\mathrm{c}\right)\int \Delta_\varphi(x)\,dN(x)\\
=&\varepsilon \int \left[\Phi(x) - \varphi(x)\right]\left[U_g[g(x)] - U_g(g_\mathrm{c})\right]\mathbb{I}_{\{g(x)<g_\mathrm{c}\}}\\
-&\varepsilon \int \left[\Phi(x) + \varphi(x)\right]\left[U_g[g(x)] - U_g(g_\mathrm{c})\right]\mathbb{I}_{\{g(x)>g_\mathrm{c}\}}.
\end{aligned}
$$

Optimality implies that the sum of these two terms is negative. But each integrand is positive by construction. Hence, each integrand is equal to zero N almost everywhere. In other words, we obtain that (10) must hold N almost everywhere. Similarly, in the OTC market, let

$$
\begin{aligned}
\hat{\gamma}(x,x') &= \gamma(x,x') + \varepsilon\left[\Gamma(x,x') - \gamma(x,x')\right]\mathbb{I}_{\{g(x)<g(x')\}} - \varepsilon\left[\Gamma(x',x) + \gamma(x,x')\right]\mathbb{I}_{\{g(x)>g(x')\}}\\
&\equiv \gamma(x,x') + \varepsilon\Delta_\gamma(x,x').
\end{aligned}
$$

Therefore:

$$
\begin{aligned}
&\frac{L(\hat{\gamma},\varphi) - L(\gamma,\varphi)}{N(X_\mathrm{o})} = \varepsilon \int U_g\left[g(x)\right]\int \Delta_\gamma(x,x')\,dN(x'\,|\,\mathrm{o})\,dN(x\,|\,X_\mathrm{o})\\
=&\frac{\varepsilon}{2}\int\int U_g\left[g(x)\right]\Delta_\gamma(x,x')\,dN(x\,|\,\mathrm{o})\,dN(x'\,|\,\mathrm{o})\\
&+\frac{\varepsilon}{2}\int\int U_g\left[g(x')\right]\Delta_\gamma(x',x)\,dN(x'\,|\,\mathrm{o})\,dN(x\,|\,\mathrm{o})\\
=&\frac{\varepsilon}{2}\int\int \left\{U_g\left[g(x)\right] - U_g\left[g(x')\right]\right\}\Delta_\gamma(x,x')\,dN(x\,|\,\mathrm{o})\,dN(x'\,|\,\mathrm{o})\\
=&\frac{\varepsilon}{2}\int\int \left\{U_g\left[g(x)\right] - U_g\left[g(x')\right]\right\}\left[\Gamma(x,x') - \gamma(x,x')\right]\mathbb{I}_{\{g(x)<g(x')\}}\,dN(x\,|\,\mathrm{o})\,dN(x'\,|\,\mathrm{o})\\
&-\frac{\varepsilon}{2}\int\int \left\{U_g\left[g(x)\right] - U_g\left[g(x')\right]\right\}\left[\Gamma(x',x) + \gamma(x,x')\right]\mathbb{I}_{\{g(x)>g(x')\}}\,dN(x\,|\,\mathrm{o})\,dN(x'\,|\,\mathrm{o}).
\end{aligned}
$$

If $\gamma(x,x')$ is optimal, this must be negative. Since both integrands are positive, they must be zero N almost everywhere. In other words, (8) holds N almost everywhere.

A.1.2 Proof of equilibrium existence

Step 1: the partial equilibrium determination of post-trade exposures. To establish existence, we start from a solution of the planner's problem and modify it so that optimality conditions hold everywhere instead of almost everywhere. Economically, this amounts to determine the post-trade exposures of any (ω, k) for any participation decision π. Formally, we consider some arbitrary $x \in X$, and we fix some arbitrary centralized market price, $U_g(h_\mathrm{c})$, and any arbitrary function for the post-trade exposures, $h(x')$, of other banks in the OTC market. Taken together, these fully determine expectations about terms of trades in both markets. We then seek a solution to the problem:

$$g = \omega(x) + \int \gamma(x, x')\, dN(x' \,|\, \mathrm{o}) + \varphi(x), \tag{17}$$

where centralized and OTC market trades $\varphi(x)$ and $\gamma(x, x')$ are chosen optimally given h_c and $h(x')$. That is:

$$\varphi(x) = \begin{cases} \Phi(x) & \text{if } g < h_\mathrm{c} \\ \in [-\Phi(x), \Phi(x)] & \text{if } g = h_\mathrm{c} \\ -\Phi(x) & \text{if } g > h_\mathrm{c} \end{cases} \tag{18}$$

Likewise:

$$\gamma(x, x') = \begin{cases} \Gamma(x, x') & \text{if } g < h(x') \\ \in [-\Gamma(x', x), \Gamma(x, x')] & \text{if } g = h(x') \\ -\Gamma(x', x) & \text{if } g > h(x'). \end{cases} \tag{19}$$

The main result is that:

Lemma 8. *The fixed point problem (17)-(19) has a unique solution. Moreover, this solution remains the same:*

- *If $h(x')$ is changed but remains the same $N(\,\cdot\,|\,\mathrm{o})$-almost everywhere;*
- *If the OTC optimality conditions (19) only hold $N(\,\cdot\,|\,\mathrm{o})$-almost everywhere.*

Step 2: modifiying a solution of the planner's problem into an equilibrium. As noted in the text, the planner's problem only determines trades and post-trade exposures N-almost everywhere, that is, on the participation path. The next step is to construct an equilibrium by modifying a solution to the planner's problem so that trading decisions are optimal off the participation path:

Lemma 9. *Let (γ, φ, g) denote a solution to the planner's problem. Then there is some $(\hat{\gamma}, \hat{\varphi}, \hat{g})$, equal to (γ, φ, g) N-almost everywhere, that is the basis of an equilibrium.*

The proof builds on Lemma 8 which allows to determine trades and post-trade exposure for all participation decisions, given aggregate market conditions. The details are in online Appendix B.5

A.1.3 Properties of the equilibrium post-trade exposures

A useful result about equilibrium post-trade exposures, $g(x)$, is that they are equi-continuous in $x \in X$.

Lemma 10. *Suppose that the trading capacity constraints $\Gamma(x, x')$ and $\Phi(x)$ are continuous. Then, there exists a positive and continuous function $G(x_2, x_1)$, satisfying $G(x, x) = 0$ for all x, and such that, for all x_1 and x_2: $|g(x_1) - g(x_2)| \leq G(x_2, x_1)$.*

What remains to show is the following lemma.

Lemma 11. *In symmetric economies, the equilibrium post-trade exposures, g, have following properties:*

- *Symmetric: For $\omega \leq \frac{1}{2}$, $g(\omega, k, \pi) = 1 - g(1-\omega, k, \pi)$.*
- *Increasing in endowment: $\omega < \omega' \Rightarrow g(\omega, k, \pi) \leq g(\omega', k, \pi)$.*
- *Less than $\frac{1}{2}$ for $\omega \leq \frac{1}{2}$: $g(\omega, k, \pi) \leq \frac{1}{2}$.*
- *Greater than $\frac{1}{2}$ for $\omega > \frac{1}{2}$: $g(\omega, k, \pi) \geq \frac{1}{2}$.*
- *Increasing in capacity for $\omega \leq \frac{1}{2}$: $k < k' \Rightarrow g(\omega, k, \pi) \leq g(\omega, k', \pi)$.*
- *Decreasing in capacity for $\omega > \frac{1}{2}$: $k < k' \Rightarrow g(\omega, k, \pi) \geq g(\omega, k', \pi)$.*
- *Increasing in centralized market participation for $\omega \leq \frac{1}{2}$: $g(\omega, k, \mathrm{o}) \leq g(\omega, k, \mathrm{oc})$.*
- *Decreasing in centralized market participation for $\omega > \frac{1}{2}$: $g(\omega, k, \mathrm{o}) \geq g(\omega, k, \mathrm{oc})$.*

The proofs of Lemma 10 and 11 are in the online Appendix B.6 and B.7.

A.2 Proof of Lemma 1

Define the Marginal Private Value, MPV(x), as the certainty equivalent payoff of a bank of type x net of the autarky value and before the participation cost. Then, (7) implies

$$\begin{aligned}
\mathrm{MPV}(x) =& U\left[g(x)\right] - U\left[\omega(x)\right] - \int \gamma(x,x')P_{\mathrm{o}}(x,x')\, dN(x' \,|\, \mathrm{o}) - \varphi(x)P_{\mathrm{c}} \\
=& U\left[g(x)\right] - U\left[\omega(x)\right] - \int \gamma(x,x')\frac{U_g\left[g(x)\right] + U_g\left[g(x')\right]}{2}\, dN(x' \,|\, \mathrm{o}) - \varphi(x)P_{\mathrm{c}} \\
=& U\left[g(x)\right] - U\left[\omega(x)\right] - \varphi(x)P_{\mathrm{c}} - \int \gamma(x,x')U_g\left[g(x')\right]\, dN(x' \,|\, \mathrm{o}) \\
& - \int \gamma(x,x')\frac{U_g\left[g(x)\right] - U_g\left[g(x')\right]}{2}\, dN(x' \,|\, \mathrm{o}) \\
=& \overbrace{U\left[g(x)\right] - U\left[\omega(x)\right] - \varphi(x)P_{\mathrm{c}} - \int \gamma(x,x')U_g\left[g(x')\right]\, dN(x' \,|\, \mathrm{o})}^{=S(x)} \\
& - \frac{1}{2}\underbrace{\int \Gamma(x,x')\left|U_g\left[g(x)\right] - U_g\left[g(x')\right]\right|\, dN(x' \,|\, \mathrm{o})}_{=B(x)},
\end{aligned}$$

where the second equality obtains by using (9), the third equality by rearranging, and the last one by using (8), which establishes the decomposition stated in the Lemma. Next, we prove the properties of the marginal private value stated in the following Lemma, whose proof is in the online Appendix B.

Lemma 12. *The marginal private value increases with k, decreases with $\omega \in \left[0, \frac{1}{2}\right]$, and symmetrically increases with $\omega \in \left[\frac{1}{2}, 1\right]$.*

A.3 Proof of Proposition 2

Given any collection of trades (γ, φ) and any ε, we define the post-trade exposure by:

$$g(x, \gamma, \varphi, \varepsilon) \equiv \omega(x) + \int \gamma(x, x') \, \frac{dN(x') + \varepsilon dn(x')}{N(X_\mathrm{o}) + \varepsilon n(X_\mathrm{o})} + \varphi(x).$$

Social welfare before participation costs is:

$$W(\gamma, \varphi, \varepsilon) \equiv \int U\left[g(x, \gamma, \varphi, \varepsilon)\right] \left(dN(x) + \varepsilon \, dn(x)\right),$$

where the function W implicitly depends on $N + \varepsilon \, n$. The corresponding Lagrangian is

$$L(\gamma, \varphi, \varepsilon) \equiv W(\gamma, \varphi, \varepsilon) - U_g\left(\frac{1}{2}\right) \int \varphi(x) \, \left[dN(x) + \varepsilon dn(x)\right].$$

Notice that Lagrange multiplier for the centralized market resource constraint is $U_g\left(\frac{1}{2}\right)$ because any admissible reallocation must maintain symmetry. Letting Λ denote the set of trades (γ, φ) that are feasible $N + \varepsilon n$ almost everywhere, and using notation that closely follow Milgrom and Segal (2002), the planner's problem is:

$$W^\star(\varepsilon) = \sup_{(\gamma,\varphi)\in\Lambda} W(\gamma, \varphi, \varepsilon) = \sup_{(\gamma,\varphi)\in\Lambda} L(\gamma, \varphi, \varepsilon)$$

where the second equality uses the fact that the solution to the planner's problem maximizes the Lagrangian. We know from our earlier results that the planner's problem has at least a solution. That is, the maximum correpondence

$$\Lambda^\star(\varepsilon) = \{(\gamma, \varphi) \in \Lambda \, : \, L(\gamma, \varphi, \varepsilon) = W^\star(\varepsilon)\}.$$

is not empty. The rest of the proof extends arguments from Milgrom and Segal (2002), and is organized as follows. In Section B.1.1 we first show that the planner's value, $W^\star(\varepsilon)$, is right-hand differentiable at $\varepsilon = 0$. In Section B.1.2, we establish that the right-hand derivative maximizes marginal social value with respect to all socially optimal trades, $(\gamma, \varphi) \in \Lambda^\star(0)$:

$$\frac{dW^\star}{d\varepsilon}(0^+) = \max_{(\gamma^\star,\varphi^\star)\in\Lambda^\star(0)} \frac{\partial L}{\partial \varepsilon}(\gamma^\star, \varphi^\star, 0). \tag{20}$$

The maximization problem on the right-hand side of (33) is not trivial because the set $\Lambda^\star(0)$ of socially optimal trades can be large. Indeed, these trades are not determined for entrants in the OTC market, that is, for types which have positive measure according to n but not to N. In Section B.1.3, we find the solution to this maximization problem: we show that it is solved by equilibrium trades. That is, to calculate the marginal social value, one should assume that the entrants in the OTC market follow their equilibrium trade.

We end this section with an explicit calculation of the partial derivative of L evaluated at equilibrium trade:

$$\frac{\partial L}{\partial \varepsilon}(\gamma, \varphi, 0) = \int U\left[g(x)\right]\, dn(x) + \int U_g\left[g(x)\right] \frac{\partial g}{\partial \varepsilon}(x)\, dN(x) - U_g\left(\frac{1}{2}\right) \int \varphi(x)\, dn(x).$$

Using that

$$\frac{\partial g}{\partial \varepsilon}(x) = \int \gamma(x, x')\, \frac{dn(x')N(X_{\mathrm{o}}) - dN(x')n(X_{\mathrm{o}})}{\left[N(X_{\mathrm{o}}) + \varepsilon\, n(X_{\mathrm{o}})\right]^2},$$

the second term can be simplified as follows:

$$\begin{aligned}
&\int\int U_g\left[g(x)\right]\gamma(x, x')\frac{dn(x')N(X_{\mathrm{o}}) - dN(x')n(X_{\mathrm{o}})}{N(X_{\mathrm{o}})^2}\, dN(x) \\
=&\int\int U_g\left[g(x')\right]\gamma(x', x)\frac{dN(x')}{N(X_{\mathrm{o}})}\, dn(x) - n(X_{\mathrm{o}})\int\int\ U_g\left[g(x)\right]\gamma(x, x')\frac{dN(x)\, dN(x')}{N(X_{\mathrm{o}})^2} \\
=&-\int\int U_g\left[g(x')\right]\gamma(x, x')\frac{dN(x')}{N(X_{\mathrm{o}})}\, dn(x) - n(X_{\mathrm{o}})\int\int U_g\left[g(x)\right]\gamma(x, x')\frac{dN(x)\, dN(x')}{N(X_{\mathrm{o}})^2}
\end{aligned} \tag{21}$$

where the first equality follows by exchanging the name of variables, namely replacing x by x' in the first term, and the second equality follows by bilateral feasibility, i.e. $\gamma(x', x) = -\gamma(x, x')$. The first term of (21) can be rewritten as follows:

$$\begin{aligned}
&-\int\int P_{\mathrm{o}}(x, x')\gamma(x, x')dN(x' \,|\, \mathrm{o})dn(x) + \int\int \left(P_{\mathrm{o}}(x, x') - U_g\left[g(x')\right]\right)\gamma(x, x')\, dN(x' \,|\, \mathrm{o})dn(x) \\
=&-\int\int P_{\mathrm{o}}(x, x')\gamma(x, x')dN(x'|\mathrm{o})dn(x) + \frac{1}{2}\int\int \left(U_g\left[g(x)\right] - U_g\left[g(x')\right]\right)\gamma(x, x')dN(x'|\mathrm{o})dn(x) \\
=&-\int\int P_{\mathrm{o}}(x, x')\gamma(x, x')dN(x'|\mathrm{o})dn(x) + \frac{1}{2}\int\int \left|U_g\left[g(x)\right] - U_g\left[g(x')\right]\right|\Gamma(x, x')dN(x'|\mathrm{o})dn(x) \\
=&-\int\int P_{\mathrm{o}}(x, x')\gamma(x, x')\, dN(x' \,|\, \mathrm{o})\, dn(x) + \frac{1}{2}\int B(x)\, dn(x),
\end{aligned}$$

where: the first line obtains by using the definition of $N(x \,|\, \mathrm{o})$ and subtracting and adding bilateral OTC payments, $P_{\mathrm{o}}(x, x')\gamma(x, x')$; the second line by substituting the expression (9) for $P_{\mathrm{o}}(x, x')$; the third line by using the optimality condition for bilateral trades (8); and the fourth line by using the definition of the bargaining surplus.

The second term of (21) can be simplified as follows:

$$
\begin{aligned}
&= -\frac{n(X_{\mathrm{o}})}{2}\int\int U_g\left[g(x)\right]\gamma(x,x')\,dN(x\,|\,\mathrm{o})\,dN(x'\,|\,\mathrm{o}) \\
&\quad -\frac{n(X_{\mathrm{o}})}{2}\int\int U_g\left[g(x')\right]\gamma(x',x)\,dN(x\,|\,\mathrm{o})\,dN(x'\,|\,\mathrm{o}) \\
&= -\frac{n(X_{\mathrm{o}})}{2}\int\int U_g\left[g(x)\right]\gamma(x,x')dN(x\,|\,\mathrm{o})dN(x'\,|\,\mathrm{o}) \\
&\quad + \quad \frac{n(X_{\mathrm{o}})}{2}\int\int U_g\left[g(x')\right]\gamma(x,x')\,dN(x\,|\,\mathrm{o})\,dN(x'\,|\,o) \\
&= -\frac{n(X_{\mathrm{o}})}{2}\int\int \left(U_g\left[g(x)\right]-U_g\left[g(x')\right]\right)\gamma(x,x')\,dN(x\,|\,\mathrm{o})\,dN(x'\,|\,\mathrm{o}) \\
&= -\frac{n(X_{\mathrm{o}})}{2}\int\int \left|U_g\left[g(x)\right]-U_g\left[g(x')\right]\right|\Gamma(x,x')\,dN(x\,|\,\mathrm{o})\,dN(x'\,|\,\mathrm{o}) \\
&= -\frac{1}{2}\int_{X_{\mathrm{o}}}\bar{B}\,dn(x).
\end{aligned}
$$

where: the first equality follows by using the definition of $N(x\,|\,\mathrm{o})$, breaking the integral into two identical halves and exchanging the name of variables in the second term, replacing x by x' in the second half; the second equality follows by bilateral feasibility $\gamma(x',x) = -\gamma(x,x')$; the third equality by collecting terms; the fourth equality using the optimality condition (8) for bilateral trades; and the fifth equality by using the definition of the average bargaining surplus.

The formula of the Proposition follows from collecting all terms, using the definition of the certainty equivalent value in (11) and its expression in terms of MPV, and by re-defining function $\varepsilon \mapsto W^{\star}\left[N + \varepsilon\left(n^{+} - n^{-}\right)\right]$ to include participation costs and with a particular choice of reallocation $n = n^{+} - n^{-}$.

A.4 Proof of Lemma 2

Post-trade exposures in the OTC market. The proof that post-trade exposures are equalized proceeds as usual by guess and verify. Namely, if the atom property holds and post-trade exposures of $\omega = 0$ banks are equal to some $\bar{g}_{\mathrm{o}} < 1/2$, then by symmetry the post-trade exposures of $\omega = 1$ banks are strictly above $1/2$. Hence, in any meeting between an $\omega = 0$ bank of capacity k and an $\omega = 1$ bank of capacity k', the $\omega = 0$ bank purchase $(k + k')/2$ assets from the $\omega = 1$ bank. Hence, the post-trade exposure of $\omega = 0$ banks with capacity k writes:

$$
\bar{g}_{\mathrm{o}} = \frac{1}{2}\int \gamma(k,k')\,dN(k'\,|\,\mathrm{o}) + \frac{1}{2}\int \frac{k+k'}{2}\,dN(k'\,|\,\mathrm{o}), \tag{22}
$$

where the first term is the net trade with other $\omega = 0$ banks and the second term is the net trade with $\omega = 1$ banks. Integrating across all k in the OTC market and keeping in mind that the aggregate net trade between all $\omega = 0$ banks must be zero, we obtain that the atom is given by:

$$\int \bar{g}_{\mathrm{o}} \, dN(k \mid \mathrm{o}) = \bar{g}_{\mathrm{o}} = \frac{\mathbb{E}\left[k' \mid \mathrm{o}\right]}{2}. \tag{23}$$

Conversely, suppose that $\mathbb{E}\left[k' \mid \mathrm{o}\right] < 1$. One can directly verify that the bilateral trades $\gamma(k, k') = (k' - k)/2$ satisfy the capacity constraint and make the post-trade exposures of all $\omega = 0$ equal to $\bar{g}_{\mathrm{o}}$ given by (23). Since $\omega = 0$ banks equalize their exposures, any feasible trade size is optimal.

Next suppose that $\mathbb{E}\left[k' \mid \mathrm{o}\right] \geq 1$. Then let $\alpha \equiv \mathbb{E}\left[k' \mid \mathrm{o}\right]^{-1} \leq 1$. Then one can directly verify that if an $\omega = 0$ bank of capacity k trades $\alpha(k + k')/2$ with $\omega = 1$ banks of capacity k', and $\alpha(k' - k)$ with $\omega = 0$ banks of capacity k', then its post-trade exposure is equal to $1/2$. Since $\alpha \leq 1$, these trades satsify the capacity constraint. Since all banks equalize their exposures to $1/2$, any feasible trade size is optimal.

Marginal private values in the OTC market. To calculate the MPV, consider first the full appropriation surplus:

$$\begin{aligned} S(0, k, \mathrm{o}) =& U(\bar{g}_{\mathrm{o}}) - U(0) \\ & - \frac{1}{2} \int U_g(\bar{g}_{\mathrm{o}}) \gamma(k, k') \, dN(k' \mid \mathrm{o}) - \frac{1}{2} \int \frac{k + k'}{2} U_g(1 - \bar{g}_{\mathrm{o}}) \, dN(k' \mid \mathrm{o}). \end{aligned}$$

Then, subtracting half of the bargaining surplus:

$$\begin{aligned} \mathrm{MPV}(0, k, \mathrm{o}) =& U(\bar{g}_{\mathrm{o}}) - U(0) \\ & - \frac{1}{2} \int U_g(\bar{g}_{\mathrm{o}}) \gamma(k, k') \, dN(k' \mid \mathrm{o}) - \frac{1}{2} \int \frac{k + k'}{2} \frac{U_g(\bar{g}_{\mathrm{o}}) + U_g(1 - \bar{g}_{\mathrm{o}})}{2} \, dN(k' \mid \mathrm{o}). \end{aligned}$$

Subtracting and adding $U_g(\bar{g}_{\mathrm{o}})\bar{g}_{\mathrm{o}}$ and using formula (22) for $\bar{g}_{\mathrm{o}}$, we obtain:

$$\begin{aligned} \mathrm{MPV}(0, k, \mathrm{o}) &= U(\bar{g}_{\mathrm{o}}) - U(0) - U_g(\bar{g}_{\mathrm{o}})\bar{g}_{\mathrm{o}} + \frac{1}{2} \int \frac{U_g(\bar{g}_{\mathrm{o}}) - U_g(1 - \bar{g}_{\mathrm{o}})}{2} \frac{k + k'}{2} \, dN(k' \mid \mathrm{o}) \\ &= \frac{|U_{gg}|}{2} \bar{g}_{\mathrm{o}}^2 + \frac{1}{2} |U_{gg}| \frac{1 - \bar{g}_{\mathrm{o}} - \bar{g}_{\mathrm{o}}}{2} \frac{k + \mathbb{E}\left[k \mid \mathrm{o}\right]}{2}, \end{aligned}$$

which simplifies to the formula shown in the Lemma after replacing $\bar{g}_{\mathrm{o}} = \mathbb{E}\left[k' \mid \mathrm{o}\right]/2$, if less than $1/2$, and otherwise by setting $\bar{g}_{\mathrm{o}} = 1/2$.

Post-trade exposure in the centralized market. Given a price of $U_g(1/2)$, it is clear that the optimal trade of an $\omega = 0$ bank in the centralized market is the smaller of $(k + K)/2$ and $1/2$.

Marginal private values in the centralized market. The marginal private value is calculated as:

$$\begin{aligned}
\mathrm{MPV}(0,k,\mathrm{c}) &= U(g(0,k,\mathrm{c})) - U(0) - U_g(1/2)g(0,k,\mathrm{c}) \\
&= U(g(0,k,\mathrm{c})) - U(0) - U_g(g(0,k,\mathrm{c}))g(0,k,\mathrm{c}) + \left(U_g(g(0,k,\mathrm{c})) - U_g(1/2)\right)g(0,k,\mathrm{c}) \\
&= \frac{|U_{gg}|}{2}g(0,k,\mathrm{c})^2 + |U_{gg}|\left(\frac{1}{2} - g(0,k,\mathrm{c})\right)g(0,k,\mathrm{c}),
\end{aligned}$$

which simplifies to the formula of the Lemma.

A.5 Proof of Lemma 3

We first need to show that $k^\star < 1-K$. This follows because, when $k = 1-K$, then a bank can attain perfect risk-sharing in the centralized market, $g_\mathrm{c}(1-K) = 1/2$, which one can verifies makes the centralized market a strictly better choice than the OTC market.

Next, we need to show that $\bar{g}_\mathrm{o} > (k^\star + K)/2$. Given our maintained assumption that there is imperfect risk-sharing, $\bar{g}_\mathrm{o} < 1/2$, we have that $\mathrm{MPV}(0,k^\star,\mathrm{o}) < |U_{gg}|/8$, the full risk-sharing MPV. Since, in addition, $\mathrm{MPV}(0,k^\star,\mathrm{o}) = \mathrm{MPV}(0,k^\star,\mathrm{c})$, it follows that risk-sharing is imperfect in the centralized market as well. Hence, $g(\omega,k^\star,\mathrm{c}) = (k+K)/2$. Using the indifference condition again, we obtain:

$$\begin{aligned}
&\mathrm{MPV}(0,k^\star,\mathrm{o}) = \mathrm{MPV}(0,k^\star,\mathrm{c}) \\
\Leftrightarrow &2\bar{g}_\mathrm{o} + k^\star(1-2\bar{g}_\mathrm{o}) = (k^\star+K)\left(2-(k^\star+K)\right) \\
\Leftrightarrow &2\bar{g}_\mathrm{o}\left(1-k^\star)\right) = (k^\star+K)\left(2-(k^\star+K)\right) - k \\
\Leftrightarrow &\left(2\bar{g}_0 - (k^\star+K)\right)(1-k^\star) = (k^\star+K)\left(2-(k^\star+K)\right) - k^\star - (1-k^\star)(k^\star+K) \\
\Leftrightarrow &\left(2\bar{g}_0 - (k^\star+K)\right)(1-k^\star) = K(1-(k^\star+K)) > 0,
\end{aligned}$$

which implies that $\bar{g}_\mathrm{o} > (k^\star+K)/2$.

A.6 Proof of Proposition 3

When $\overline{k} \leq 1$. Then there are no banks with $k > 1$ in the OTC market and so follows from Lemma 3 that:

$$\bar{g}_\mathrm{o} = \frac{1}{2}\min\left\{\mathbb{E}\left[k' \,|\, k' \leq k^\star\right], 1\right\} \leq \frac{k^\star}{2}.$$

According to Lemma 3, if there were positive participation in both market, then we would have that $\bar{g}_\mathrm{o} > k^\star/2$. Hence, if $\overline{k} \leq 1$, there does not exist an equilibrium with positive participation in both markets.

When $\overline{k} > 1$: equilibrium such that $\bar{g}_{\mathbf{o}} < 1/2$. In this paragraph, we show that, there exists an equilibrium with $\bar{g}_{\mathrm{o}} < 1/2$ if and only if $\mathbb{E}\left[k' \,|\, k' \leq 1-K \text{ and } k' \geq 1\right] < 1$, and that this equilibrium is unique.

It is clear that, if an equilibrium with $\bar{g}_{\mathrm{o}} < 1/2$ exists, then

$$\bar{g}_{\mathrm{o}} = \hat{g}(k^\star) < \frac{1}{2} \text{ where } \hat{g}(x) \equiv \frac{1}{2}\mathbb{E}\left[k' \,|\, k' \leq x \text{ or } k' \geq 1\right].$$

Taking derivatives, we find that $\hat{g}'(x)$ is proportional to:

$$\begin{aligned} h(x) &= x\left(\int_{\underline{k}}^{x} f(k)\,dk + \int_{1}^{\overline{k}} f(x)\,dk\right) - \left(\int_{\underline{k}}^{x} kf(k)\,dk + \int_{1}^{\overline{k}} kf(x)\,dk\right) \\ &= \mathbb{P}\left(k \leq x \text{ or } k \geq 1\right) \times \left(x - \mathbb{E}\left[k \,|\, k \leq k \text{ and } x \geq 1\right]\right). \end{aligned} \tag{24}$$

Moreover, $h'(x) = F(x) + 1 - F(1) > 0$. Therefore, $h(\underline{k}) < 0$ and $h(x)$ changes sign at most once.

With this calculation in mind we show that, if an equilibrium with $\bar{g}_{\mathrm{o}} < 1/2$ exists, then

$$\mathbb{E}\left[k' \,|\, k' \leq 1-K \text{ or } k' \geq 1\right] < 1.$$

Indeed, if the opposite inequality holds, then one easily verifies that $\hat{g}(1-K) \geq 1/2$ and $h(1-K) \leq 0$. Moreover, since $h'(x) > 0$, it follows that $h(x) \leq 0$ for all $x \leq 1-K$. Hence, $\hat{g}(x)$ is decreasing over $[\underline{k}, 1-K]$ and $\hat{g}(x) \geq 1/2$ over $[\underline{k}, 1-K]$. But we have reached a contradiction because we have shown that, in an equilibrium with $\bar{g}_{\mathrm{o}} < 1/2$, $\bar{g}_{\mathrm{o}} = \hat{g}(k^\star) < 1/2$.

Next, we show the converse: if $\mathbb{E}\left[k' \,|\, k' \leq 1-K \text{ or } k' \geq 1\right] < 1$, then there exists an equilibrium with $\bar{g}_{\mathrm{o}} < 1/2$. Indeed, evaluating the equilibrium equation at $k^\star = \underline{k}$ and keeping in mind that $\overline{k} > 1$, we obtain that $\hat{g}(\underline{k}) > 1/2$, which implies that $\mathrm{MPV}(\underline{k} \,|\, \hat{g}(\underline{k})\,\mathrm{o}\,) > \mathrm{MPV}(\underline{k} \,|\, \mathrm{c}\,)$. At $k^\star = 1-K$, on the other hand, since $\hat{g}(1-K) < 1/2$, one can directly verify that $\mathrm{MPV}(1-K \,|\, \hat{g}(1-K), \mathrm{o}\,) < \mathrm{MPV}(1-K \,|\, \mathrm{c}\,)$. Taken together, this means that the equilibrium equation has a unique solution $k^\star < 1-K$. Since, at a solution, the marginal bank obtains imperfect risk sharing in the centralized market, we must have that $\hat{g}(k^\star) < 1/2$.

Finally, we show that the equilibrium just identified must be unique. Indeed, differentiating the equilibrium equation (15) with respect to $k^\star$, we then obtain:

$$\frac{\partial \mathrm{MPV}}{\partial k^\star}(0, k^\star, \mathrm{o} \,|\, \hat{g}(k^\star)) - \frac{\partial \mathrm{MPV}}{\partial k^\star}(0, k^\star, \mathrm{c}) + \frac{\partial \mathrm{MPV}}{\partial \bar{g}_{\mathrm{o}}}(0, k^\star, \mathrm{o} \,|\, \hat{g}(k^\star)\,)\frac{d\hat{g}}{dk^\star}(k^\star).$$

The difference between first two terms is strictly negative. Recall indeed that, holding $\bar{g}_{\mathrm{o}} = \hat{g}(k^\star)$ constant, $k^\star$ is the smallest of two intersections between two functions: $\mathrm{MPV}(0, k, \mathrm{o})$, which is linear and strictly increasing in k, and $\mathrm{MPV}(0, k, \mathrm{c})$, which is strictly concave. Hence, it must be the case that the first function crosses the second from above. The third term is also negative. Indeed, since $k^\star \leq 1$, the partial derivative of the MPV with respect to $\bar{g}_{\mathrm{o}}$ is positive. From Lemma 3, we have that $k^\star/2 \leq \hat{g}(k^\star)$ or, equivalently, that $k^\star \leq \mathbb{E}\,[k \,|\, k \leq k^\star \text{ or } k \geq 1]$. It follows from equation (24) that $d\hat{g}/dk^\star(k^\star) \leq 0$. This establishes the claim.

Equilibrium such that $\bar{g}_{\mathrm{o}} = 1/2$. We already know that, for such an equilibrium to exist, we must have $\mathbb{E}\,[k' \,|\, k' \leq 1-K \text{ or } k' \geq 1] \geq 1$. Moreover, if such an equilibrium exists, then $\mathrm{MPV}(k \,|\, \mathrm{o}\,) = |U_{gg}|/8$, which is strictly greater than $\mathrm{MPV}(k \,|\, \mathrm{c}\,)$ for $k < 1-K$, and equal to $\mathrm{MPV}(k \,|\, \mathrm{c}\,)$ for $k \geq 1-K$. Therefore the following banks participate in the OTC markets: all banks with capacities $k < 1-K$ participate in the OTC market, and some measurable subset B of banks in $[1-K, \overline{k}]$. Next, we verify that we can pick some measurable $B \subseteq [1-K, \overline{k}]$ such that $\bar{g}_{\mathrm{o}} = 1/2$. From Lemma 2, this is equivalent to:

$$\mathbb{E}\left[k' \,|\, k' \leq 1-K \text{ and } k' \in B\right] \geq 1,$$

for some measurable $B \subseteq [1-K, \overline{k}]$. Equivalently:

$$\int_{\underline{k}}^{1-K} (k-1)\, f(k)\, dk + \int_{k \in B} (k-1)\, f(k)\, dk \geq 0.$$

Clearly, the second term is maximized if $B = [1, \overline{k}]$, in which case the inequality simplifies to our maintained assumption that $\mathbb{E}\,[k' \,|\, k' \leq 1-K \text{ or } k' \geq 1] \geq 1$. Hence, we can find some measurable $B \in [1-K, \overline{k}]$ such that $\bar{g}_{\mathrm{o}} = 1/2$.

A.7 Proof of Lemma 4

Assume that $C(\mathrm{o}) \simeq C(\mathrm{c})$. Then, the analysis of post-trade exposures and optimal participation patterns is the same as before. Assuming that $\bar{g}_{\mathrm{o}} < 1/2$, the equilibrium equation can be written

$$H(k) - \Delta = 0$$

where

$$\begin{aligned} H(k) &\equiv \frac{1}{8}\left(2\hat{g}(k) + k\left(1 - 2\hat{g}(k)\right)\right) - \frac{1}{2} g(0,k,\mathrm{c})\left(1 - g(0,k,\mathrm{c})\right) \\ \Delta &\equiv \frac{C(\mathrm{o}) - C(\mathrm{c})}{|U_{gg}|}. \end{aligned}$$

If $\mathbb{E}\,[k' \,|\, k' < 1-K \text{ and } k' \geq 1] < 1$ and $\Delta = 0$, we know from Proposition 3 that the equilibrium equation has two solutions which we denoted by $k^\star(0) < 1-K$ and $k^{\star\star}(0) = 1$. Going through th same steps as in the proof of 3, it is immediate to show that, when $\Delta \simeq 0$, then there are also two solutions: a unique solution $k^\star(\Delta)$ in the interval $(\underline{k}, 1-K)$, and a unique solution $k^{\star\star}(\Delta)$ in the interval $(1-K, \overline{k})$.

The proof of existence established that $H'(k^\star(0)) < 0$. Differentiating at $k^{\star\star}(0) = 1$, one can verify that $H'(k^{\star\star}(0)) > 0$ since $\bar{g}_\mathrm{o} < 1/2$. An application of the Implicit Function Theorem shows that, for $\Delta \simeq 0$, $k^\star(\Delta)$ is increasing in Δ and $k^{\star\star}(\Delta)$ is decreasing in Δ. The result follows.

A.8 Proof of Proposition 4

Preliminary result: the social planner's objective. With symmetric participation, the planner's objective can be written:

$$W(N) = \frac{1}{2}\int [U(0) + U(1)]\, dN(k,\mathrm{a}) + \frac{1}{2}\int [U(\bar{g}_\mathrm{o}) + U(1-\bar{g}_\mathrm{o}) - 2C]\, dN(k,\mathrm{o}) \\ + \frac{1}{2}\int [U(g_\mathrm{c}(k)) + U(1-g_\mathrm{c}(k)) - 2C]\, dN(k,\mathrm{c}), \tag{25}$$

where $\bar{g}_\mathrm{o}$ and $g_\mathrm{c}(k)$ are given by Lemma 2.

Preliminary results: the marginal social values. Direct calculations show that the marginal social values are:

$$\mathrm{MSV}(0,k,\mathrm{o}) = \frac{|U_{gg}|}{8}\left(2k\,(1-2\bar{g}_\mathrm{o}) + 4\bar{g}_\mathrm{o}^2\right)$$
$$\mathrm{MSV}(0,k,\mathrm{c}) = \frac{|U_{gg}|}{2} g(0,k,\mathrm{c})\,(1-gg(0,k,\mathrm{c}))\,.$$

If C is small enough, all banks participate in some market. To show this claim suppose that some banks choose autarky and consider the participation path obtained by moving all banks in autarky to the centralized market:

$$d\hat{N}(k,\mathrm{a}) = 0, d\hat{N}(k,\mathrm{o}) = dN(k,\mathrm{o}),\ \text{and } d\hat{N}(k,\mathrm{c}) = dN(k,\mathrm{c}) + dN(k,\mathrm{a}),$$

The associated change in welfare is:

$$W(\hat{N}) - W(N) = \frac{1}{2}\int [U(g(0,k,\mathrm{c})) - U(0) + U(1-g(0,k,\mathrm{c})) - U(1) - 2C]\, dN(k,\mathrm{a}) \\ = \frac{1}{2}\int [|U_{gg}|g(0,k,\mathrm{c})\,(1-g(0,k,\mathrm{c})) - 2C]\, dN(k,\mathrm{a}),$$

where the second line follows from direct calculations. Since $g(0,k,\mathrm{c}) \geq K/2$, the result follows.

Without loss, banks with $k \geq 1$ participate in the OTC market. If there is no participation in the OTC market, then consider the following alternative participation path. All banks with $k \geq 1$ participate in the OTC market, and all bank with $k < 1$ participate in the centralized market. The $k \geq 1$ banks who now participate in the OTC market have the same payoff as in the centralized market: their post-trade exposure is $\bar{g}_{\text{o}} = 1/2$. The $k < 1$ banks who participate in the centralized market keep the same post-trade exposure as well. Hence, social welfare is the same.

If there is some participation in the OTC market, then a marginal increase in the participation of a $k > 1$ bank in the OTC market create a marginal change in social welfare equal to:

$$\begin{aligned}\text{MSV}(0,k,\text{o}) - \text{MSV}(0,k,c) &= \frac{|U_{gg}|}{8}\left(2k\left(1-2\bar{g}_{\text{o}}\right) + 4\bar{g}_{\text{o}}^2 - 1\right) \\ &= \frac{|U_{gg}|}{8}\left(1-2\bar{g}_{\text{o}}\right)\left[2k - \left(1+2\bar{g}_{\text{o}}\right)\right] \geq 0,\end{aligned}$$

since $k \geq 1$ and $\bar{g}_{\text{o}} \leq 1/2$. Hence, increasing participation of $k \geq 1$ banks always increase social welfare weakly. It follows that welfare increases if we modify the participation path so that all $k \geq 1$ participate in the OTC market, but all other participation decisions remain the same.

Existence of a solution to the planner's problem. Given our previous result that it is never optimal to leave a bank in autarky, the planner's problem is to choose a participation path N to maximize social welfare (25), subject to the constraint that all banks participate in either the OTC or the centralized market, that is: $N(B,\text{o}) + N(B,\text{c}) = F(B)$ for all Borel sets B of $[\underline{k}, \overline{k}]$, and where F is the exogenous distribution of capacities in the bank population. By Theorem 14.1 in Aliprantis and Border (2006), the above constraint can be equivalently written as:

$$\int h(k)\,\left[dN(k,\text{o}) + dN(k,\text{c})\right] = \int h(k)\,dF(k), \tag{26}$$

for all continuous functions h over $[\underline{k}, \overline{k}]$. Given that we know that it is weakly optimal to make all $k \geq 1$ banks participate in the OTC market, we add the constraint:

$$\int \rho(k)\,dN(k,\text{o}) = \sum_{\pi\in\Pi}\int \rho(k)\mathbb{I}_{\{\pi=\text{o}\}}\,dN(k,\pi) = \int \rho(k)\,dF(k), \tag{27}$$

where

$$\rho(k) = \begin{cases} 0 & \text{if } k \leq 1 \\ (k-1)/\varepsilon & \text{if } k \in [1, 1+\varepsilon] \\ 1 & \text{if } k \in [1+\varepsilon, \overline{k}]. \end{cases}$$

It is clear that this constraint is satisfied for all N such that all $k \geq 1$ banks participate in the OTC market. Now, notice that the constraints (26) and (27) are defined by integrals against continuous functions. Therefore, they define a constraint set that is compact with respect to the topology of weak convergence. Moreover, for any N in the constraint set (27) holds, it follows that for any N in this constraint set:

$$\int dN(k, \mathrm{o}) \geq \int \rho(k)\, dN(k, \mathrm{o}) = \int \rho(k)\, dF(k) > 0.$$

Therefore, the function:

$$N \mapsto \mathbb{E}\left[k \,|\, \mathrm{o}\right] = \frac{\int k\, dN(k, \mathrm{o})}{\int \, dN(k, \mathrm{o})}$$

is continuous at any N in the constraint set. By implication, post-trade exposures $\bar{g}_{\mathrm{o}}$ are continuous in N, and so is the planner's objective $W(N)$. The existence of a solution then follows from the fact that the planner's problem is to maximize a continuous function over a compact constraint set.

Full risk sharing obtains if and only if $\mathbb{E}\left[k \,|\, k \leq 1 - K \text{ or } k \geq 1\right] \geq 1$. For the "if" part, suppose that the planner's problem features full risk sharing, and consider the planner's solution such that all $k \geq 1$ banks participate in the OTC market. Then, for all banks who participate in the centralized market, $g_{\mathrm{c}}(k) = 1/2$ implying that $k \geq 1 - K$. Therefore, the set of banks who participate in the OTC market is $k \in [\underline{k}, 1 - K]$ and $[1, \overline{k}]$. Since there is full risk sharing, $\bar{g}_{\mathrm{o}} = 1/2$ and

$$\mathbb{E}\left[k \,|\, k \leq 1 - K \text{ or } k \geq 1\right] \geq 1.$$

For the "only if" part, notice that, given that no bank stays in autarky, the value of the planner's problem is bounded above by that of full risk sharing, net of participation costs. The result follows because, if $\mathbb{E}\left[k \,|\, k \leq 1 - K \text{ or } k \geq 1\right] \geq 1$, then risk-sharing is attained in equilibrium, and so is feasible for the planner.

Participation threshold in the partial risk-sharing case. Direct calculation show that the marginal social values are:

$$\mathrm{MSV}(0, k, \mathrm{o}) = \frac{|U_{gg}|}{8}\left(2k\left(1 - 2\bar{g}_{\mathrm{o}}\right) + 4\bar{g}_{\mathrm{o}}^2\right)$$
$$\mathrm{MSV}(0, k, \mathrm{c}) = \frac{|U_{gg}|}{2} g(0, k, \mathrm{c})\left(1 - g(0, k, \mathrm{c})\right).$$

Proposition 2 imply that $\pi = \mathrm{o}$ if $\mathrm{MSV}(k \,|\, \mathrm{o}) > \mathrm{MSV}(k \,|\, \mathrm{c})$ and $\pi = \mathrm{c}$ if the opposite strict inequality holds.

Given partial risk sharing, $\bar{g}_{\mathrm{o}} < 1/2$ and $\mathrm{MSV}(k \,|\, \mathrm{o})$ is linear and strictly increasing in k. Moreover, when $k > 1 - K$, $\mathrm{MSV}(0, k, \mathrm{c})$ is constant and when $k \leq 1 - K$, $\mathrm{MSV}(0, k, \mathrm{c})$ is strictly concave in k. Therefore, $\mathrm{MSV}(0, k, \mathrm{o}) - \mathrm{MSV}(0, k, \mathrm{c})$ has at most two roots, implying that socially optimal participation is characterized by two thresholds as claimed. Moreover, one easily verifies that $\mathrm{MSV}(0, 1, \mathrm{o}) - \mathrm{MSV}(0, 1, \mathrm{c}) > 0$, implying that the upper threshold is strictly less than one.

A.9 Proof of Proposition 5

We maintain the assumption stated in Proposition and proceed as follows. First, given some participation threshold $k^\star < \bar{k}$ but close to $\bar{k}$, we solve for the post-trade exposures of $\pi = \mathrm{o}$ and $\pi = \mathrm{oc}$. Second, given the post-trade exposures, we solve for the MPV$'s$. Third, we verify that that the slope of $\mathrm{MPV}(k \,|\, \mathrm{oc})$ is larger than that of $\mathrm{MPV}(k \,|\, \mathrm{o})$, so that these MPV cross only once. Fourth, we set C so that the crossing occurs at $k^\star$. Let us first define

$$\begin{aligned} \bar{g}_{\mathrm{o}} &\equiv \frac{\mathbb{E}\left[k' \,|\, \mathrm{o}\right]}{2} + \frac{m_{\mathrm{oc}}}{m_{\mathrm{o}} + m_{\mathrm{oc}}} \frac{\mathbb{E}\left[k' \,|\, \mathrm{oc}\right]}{2} \\ \bar{g}_{\mathrm{oc}} &\equiv \frac{\mathbb{E}\left[k' \,|\, \mathrm{oc}\right] + K}{2} + \frac{m_{\mathrm{oc}}}{m_{\mathrm{o}} + m_{\mathrm{oc}}} \frac{\mathbb{E}\left[k' \,|\, \mathrm{oc}\right]}{2}. \end{aligned} \tag{28}$$

Then, the post-trade exposures for $\pi = \mathrm{o}$ are given by the following Lemma:

Lemma 13. *For all $k^\star$ sufficiently close to $\bar{k}$, the post trade exposures are*

$$\begin{aligned} g_{\mathrm{o}}(k) &= \bar{g}_{\mathrm{o}} \\ g_{\mathrm{oc}}(k) &= \max\left\{\bar{g}_{\mathrm{o}}, \min\left\{\bar{g}_{\mathrm{oc}}, \frac{k+K}{2} + \frac{m_{\mathrm{oc}}}{m_{\mathrm{o}} + m_{\mathrm{oc}}} \frac{k + \mathbb{E}\left[k' \,|\, \mathrm{oc}\right]}{2}\right\}\right\} \end{aligned} \tag{29}$$

In particular, $g_{\mathrm{oc}}(k) = \bar{g}_{\mathrm{oc}}$ for all $k \in [k^\star, \bar{k}]$.

Next, we study post-trade exposures in the limit $k^\star \to \bar{k}$:

Lemma 14. *As $k^\star \to \bar{k}$, the post-trade exposures admit the expansion*

$$\begin{aligned} g_{\mathrm{o}}(k) &= \bar{g}_{\mathrm{o}}^{\dagger} + o(1) \\ g_{\mathrm{oc}}(k) &= g_{\mathrm{oc}}^{\dagger}(k) + o(1), \end{aligned}$$

where the $\bar{g}_{\mathrm{o}}^{\dagger} \equiv \frac{\mathbb{E}[k]}{2}$ and $g_{\mathrm{oc}}^{\dagger}(k) \equiv \max\left\{g_{\mathrm{o}}, \min\left\{\frac{k+K}{2}, \frac{1}{2}\right\}\right\}$.

Next, consider the marginal private values:

Lemma 15. *The marginal private value of $\pi = \mathrm{o}$ is:*

$$\begin{aligned} \mathrm{MPV}(k \,|\, \mathrm{o}) = & \frac{|U_{gg}|}{2} \bar{g}_{\mathrm{o}}^2 \\ & + \frac{m_{\mathrm{o}}}{m_{\mathrm{o}} + m_{\mathrm{oc}}} \frac{|U_{gg}|}{8} \left(1 - 2\bar{g}_{\mathrm{o}}\right) \left(k + \mathbb{E}\left[k' \,|\, \mathrm{o}\right]\right) \\ & + \frac{m_{\mathrm{oc}}}{m_{\mathrm{o}} + m_{\mathrm{oc}}} \frac{|U_{gg}|}{8} \left(1 - 2\bar{g}_{\mathrm{o}}\right) \left(k + \mathbb{E}\left[k' \,|\, \mathrm{oc}\right]\right). \end{aligned}$$

The marginal private value of $\pi = \mathrm{oc}$ is:

$$\begin{aligned}\mathrm{MPV}(k \,|\, \mathrm{oc}) =& \frac{|U_{gg}|}{2} g_{\mathrm{oc}}(k)^2 + \frac{|U_{gg}|}{4} \left(1 - 2g_{\mathrm{oc}}(k)\right)(k+K) \\ &+ \frac{m_{\mathrm{o}}}{m_{\mathrm{o}} + m_{\mathrm{oc}}} \frac{|U_{gg}|}{8} \left(1 - 2\bar{g}_{\mathrm{o}}\right)\left(k + \mathbb{E}\left[k' \,|\, \mathrm{o}\right]\right) \\ &+ \frac{m_{\mathrm{oc}}}{m_{\mathrm{o}} + m_{\mathrm{oc}}} \frac{|U_{gg}|}{8} \left(1 - 2g_{\mathrm{oc}}(k)\right)\left(k + \mathbb{E}\left[k' \,|\, \mathrm{oc}\right]\right).\end{aligned}$$

As for the post-trade exposures, we can derive expansions for the marginal private values as $k^\star \to \bar{k}$. We first define:

$$\begin{aligned}\mathrm{MPV}^\dagger(k \,|\, \mathrm{o}) &\equiv \frac{|U_{gg}|}{2}(g_{\mathrm{o}}^\dagger)^2 + \frac{|U_{gg}|}{8}\left(1 - 2g_{\mathrm{o}}^\dagger\right)(k + \mathbb{E}[k]) \\ \mathrm{MPV}^\dagger(k \,|\, \mathrm{oc}) &\equiv \frac{|U_{gg}|}{2}(g_{\mathrm{oc}}^\dagger(k))^2 + \frac{k+K}{2}\frac{|U_{gg}|}{2}\left(1 - 2g_{\mathrm{oc}}^\dagger(k)\right) + \frac{|U_{gg}|}{8}\left(1 - 2g_{\mathrm{o}}^\dagger\right)(k + \mathbb{E}[k]).\end{aligned}$$

Then, we obtain:

Lemma 16. *As $k^\star \to \bar{k}$, $\mathrm{MPV}(k \,|\, \mathrm{o}) = \mathrm{MPV}^\dagger(k \,|\, \mathrm{o}) + o(1)$, $\mathrm{MPV}(k \,|\, \mathrm{oc}) = \mathrm{MPV}^\dagger(k \,|\, \mathrm{oc}) + o(1)$, and*

$$\frac{d\mathrm{MPV}}{dk}(k \,|\, \mathrm{oc}) - \frac{d\mathrm{MPV}}{dk}(k \,|\, \mathrm{o}) \geq \frac{d\mathrm{MPV}^\dagger}{dk}(k \,|\, \mathrm{oc}) - \frac{d\mathrm{MPV}^\dagger}{dk}(k \,|\, \mathrm{o}) + o(1).$$

Since $\mathrm{MPV}^\dagger(k \,|\, \mathrm{oc}) - \mathrm{MPV}^\dagger(k \,|\, \mathrm{o})$ is concave, and since our maintained assumption that $K + \bar{k} < 1$ implies that:

$$\frac{d\mathrm{MPV}^\dagger}{dk}(\bar{k} \,|\, \mathrm{oc}) - \frac{d\mathrm{MPV}^\dagger}{dk}(\bar{k} \,|\, \mathrm{o}) > 0,$$

we obtain that, for all $k^\star$ close enough to $\bar{k}$, $\mathrm{MPV}(k \,|\, \mathrm{oc}) - \mathrm{MPV}(k \,|\, \mathrm{o})$ is strictly increasing in $k \in [\underline{k}, \bar{k}]$. This means that if we pick C such that

$$C = \mathrm{MPV}(k^\star \,|\, \mathrm{oc}) - \mathrm{MPV}(k^\star \,|\, \mathrm{o}),$$

then, for all $k^\star$ close enough to $\bar{k}$, $\pi = \mathrm{o}$ is optimal for $k \in [0, k^\star)$, and $\pi = \mathrm{oc}$ is optimal for $k \in [k^\star, \bar{k}]$.

A.10 Derivation of the equilibrium of Section 4.2.2

Assume that there are thresholds $0 < k^\star < k^{\star\star} < 1$ such that the participation patterns is $[0, k^\star] = X_{\mathrm{o}} \backslash X_{\mathrm{oc}}$, $(k^\star, k^{\star\star}) = X_{\mathrm{c}} \backslash X_{\mathrm{oc}}$, and $[k^{\star\star}, 1] = X_{\mathrm{oc}}$. Given the uniform distribution for capacities, in the OTC market the masses of exclusive and non-exclusive participants are respectively

$$m_{\mathrm{o}} = k^\star, \text{ and } m_{\mathrm{oc}} = 1 - k^{\star\star}.$$

The average post-trade exposure of banks with $\omega = 0$ and capacity $k \in [0, k^\star]$, participating only in the OTC market, computes as

$$
\begin{aligned}
\bar{g}_\text{o} = & \frac{1}{2}\frac{m_\text{o}}{m_\text{oc}+m_\text{o}}\mathbb{E}\left[\frac{\max(k',k'')}{2}\middle|(k',k'')\in(X_\text{o}\backslash X_\text{oc})^2\right] \\
& + \frac{m_\text{oc}}{m_\text{oc}+m_\text{o}}\mathbb{E}\left[\frac{\max(k',k'')}{2}\middle|k'\in X_\text{o}\backslash X_\text{oc}, k''\in X_\text{oc}\right] \\
& = \frac{1}{2}\frac{m_\text{o}}{m_\text{oc}+m_\text{o}}\frac{1}{2}\frac{2k^\star}{3} + \frac{m_\text{oc}}{m_\text{oc}+m_\text{o}}\frac{1}{2}\frac{1+k^{\star\star}}{2} \\
& = \frac{1}{2}\left(\frac{m_\text{o}}{m_\text{oc}+m_\text{o}}\frac{k^\star}{3} + \frac{m_\text{oc}}{m_\text{oc}+m_\text{o}}\frac{1+k^{\star\star}}{2}\right) \\
& = \frac{1}{2}\left(\frac{k^\star}{1-k^{\star\star}+k^\star}\frac{k^\star}{3} + \frac{1-k^{\star\star}}{1-k^{\star\star}+k^\star}\frac{1+k^{\star\star}}{2}\right)
\end{aligned}
$$

In the following lemma, we derive the post trade exposure of any bank given its participation decision.

Lemma 17. *Given a participation pattern, $\{k^\star, k^{\star\star}\}$, the post trade exposures of a bank with capacity $k \in [0,1]$, and endowment $\omega = 0$, that participates exclusively in the centralized exchange or in the OTC market, or non exclusively in both markets, respectively, are*

$$
\begin{aligned}
g(0,k,\text{c}) &= g_\text{c}(k) = \frac{k}{2} \\
g(0,k,\text{o}) &= g_\text{o}(k) = \min\left[\max\left(\bar{g}_\text{o}, \frac{m_\text{oc}}{m_\text{oc}+m_\text{o}}\mathbb{E}\left[\frac{\max(k,k')}{2}\middle|k'\in X_\text{oc}\right]\right), \frac{1}{2}\right] \\
g(0,k,\text{oc}) &= g_\text{oc}(k) = \min\left[\max\left(\bar{g}_\text{o}, \frac{k}{2} + \frac{m_\text{oc}}{m_\text{oc}+m_\text{o}}\mathbb{E}\left[\frac{\max(k,k')}{2}\middle|k'\in X_\text{oc}\right]\right), \frac{1}{2}\right]
\end{aligned}
$$

Then, in next lemma, we obtain the marginal private value of different participation decisions of any bank.

Lemma 18. *Given a participation pattern, $\{k^\star, k^{\star\star}\}$, the gross marginal private value of entry, of a bank with capacity $k \in [0,1]$ and endowment $\omega = 0$, exclusively in the centralized exchange, exclusively* *in the OTC market, or non exclusively in both markets are, respectively,*

$$\text{MPV}(0,k,\text{c}) = \frac{|U_{gg}|}{2}\frac{k}{2}\left(1-\frac{k}{2}\right)$$

$$\begin{aligned}
&\text{MPV}(0,k,\text{o})\\
&\quad= \frac{|U_{gg}|}{2}\left[\frac{1}{4}-\left(\frac{1}{2}-g_{\text{o}}(k)\right)\left(\frac{1}{2}+\max\left(0,\bar{g}_{\text{o}}-\frac{m_{\text{oc}}}{m_{\text{oc}}+m_{\text{o}}}\mathbb{E}\left[\frac{\max(k,k')}{2}\middle| k'\in X_{\text{oc}}\right]\right)\right)\right]\\
&\qquad+\frac{|U_{gg}|}{2}\left(\frac{1}{2}-\bar{g}_{\text{o}}\right)\frac{m_{\text{o}}}{m_{\text{oc}}+m_{\text{o}}}\mathbb{E}\left[\frac{\max(k,k')}{2}\middle| k'\in X_{\text{o}}\backslash X_{\text{oc}}\right]
\end{aligned}$$

$$\begin{aligned}
&\text{MPV}(0,k,\text{oc})\\
&= \frac{|U_{gg}|}{2}\left[\frac{1}{4}-\left(\frac{1}{2}-g_{\text{oc}}(k)\right)\left(\frac{1}{2}-\frac{k}{2}+\max\left(0,\bar{g}_{\text{o}}-\frac{k}{2}-\frac{m_{\text{oc}}}{m_{\text{oc}}+m_{\text{o}}}\mathbb{E}\left[\frac{\max(k,k')}{2}\middle| k'\in X_{\text{oc}}\right]\right)\right)\right]\\
&\qquad+\frac{|U_{gg}|}{2}\left(\frac{1}{2}-\bar{g}_{\text{o}}\right)\frac{m_{\text{o}}}{m_{\text{oc}}+m_{\text{o}}}\mathbb{E}\left[\frac{\max(k,k')}{2}\middle| k'\in X_{\text{o}}\backslash X_{\text{oc}}\right]
\end{aligned}$$

In order to fully derive the marginal value of entries as function of k, $k^\star$, and $k^{\star\star}$, can must compute the following objects,

$$\begin{aligned}
\mathbb{E}\left[\max(k,k'))\middle| k'\in X_{\text{o}}\backslash X_{\text{oc}}\right] &= \begin{cases}\frac{(k^\star)^2+k^2}{2k^\star} \text{ if } k<k^\star\\ k \text{ if } k\geq k^\star\end{cases}\\
&= \frac{(k^\star)^2+\min(k,k^\star)^2}{2k^\star}+\max(k-k^\star,0),
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{E}\left[\max(k,k'))\middle| k'\in X_{\text{oc}}\right] &= \begin{cases}\frac{1+k^{\star\star}}{2} \text{ if } k<k^{\star\star}\\ \frac{k(k-k^{\star\star})+\frac{1-k^2}{2}}{1-k^{\star\star}} = \frac{1+k^{\star\star}}{2}+\frac{(k-k^{\star\star})^2}{2(1-k^{\star\star})} \text{ if } k^{\star\star}\leq k\leq 1\\ k \text{ if } k>1\end{cases}\\
&= \frac{1+k^{\star\star}}{2}+\frac{(\min[1,\max(k,k^{\star\star})]-k^{\star\star})^2}{2(1-k^{\star\star})}+\max(k-1,0).
\end{aligned}$$

Using previous results, an equilibrium can be pinned down by finding $0<k^\star<k^{\star\star}<1$ such that

$$\begin{aligned}
&\max\left\{\text{MPV}(0,k,\text{oc})-|U_{gg}|/16, \text{MPV}(0,k,\text{o}), \text{MPV}(0,k,\text{c})\right\}\\
&= \begin{cases}\text{MPV}(0,k,\text{o}) \text{ if } k\in[0,k^\star]\\ \text{MPV}(0,k,\text{c}) \text{ if } k\in[k^\star,k^{\star\star}]\\ \text{MPV}(0,k,\text{oc})-|U_{gg}|/16 \text{ if } k\in[k^{\star\star},1]\end{cases}
\end{aligned}$$

As we have set $C(\text{oc})=|U_{gg}|/16$, and as all the MPVs expressions are proportional to $|U_{gg}|$, one can see that the term $|U_{gg}|$ simplifies. We can therefore normalize $|U_{gg}|$ to one without loss of generality.

Mathematica finds a solution to the former equation system, $k^\star \simeq 0.202213,\ k^{\star\star} \simeq 0.932682$. Then, with *Mathematica*, in Figure 6 we use those values to plot the three (net) MPVs of entry, as functions of k, and verify that entry incentives are indeed consistent.

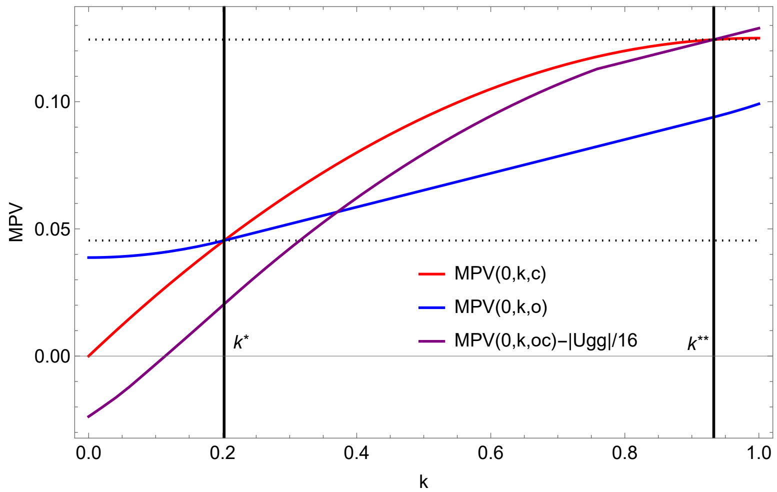

Figure 6: The net MPVs of participation in the centralized market (red curve), the OTC market (blue curve), and both markets (purple curve), as functions of capacity, k, when $k^\star = 0.202213,\ k^{\star\star} = 0.932682$.

In equilibrium, we should make sure that if a $k = 1$ bank participates exclusively in the OTC market, its post trade exposure does not reach 1/2. By taking a "directional" positions with the banks in X_{oc}, the post-trade exposure of this bank is equal to $(m_{\mathrm{oc}}/2)/(m_{\mathrm{oc}} + m_{\mathrm{o}})$, which is indeed lower. In equilibrium we should also verify that $\bar{g}_{\mathrm{o}} < 1/2$, that $g_{\mathrm{o}}(k) = \bar{g}_{\mathrm{o}}$ for $k \in [0, k^\star]$, and that $g_{\mathrm{oc}}(k) = 1/2$ for $k \in [k^{\star\star}, 1]$. With *Mathematica*, we find that $\bar{g}_{\mathrm{o}} \simeq 0.145961$ and that the equilibrium post trade exposures are consistent as shown in Figure 7.

In the proof of Lemma 18, we derived the bargaining surplus of bank as a function of its trading capacity and its participation decision. Using those result, we can compute the equilibrium bargaining surpluses. First, the bargaining surplus of a bank which participates exclusively in the OTC markets with capacity $k \in [0, k^\star]$, is equal to

$$B(0, k, \mathrm{o}) = |U_{gg}| \left(\frac{1}{2} - \bar{g}_{\mathrm{o}}\right) \mathbb{E}\left[\frac{\max(k, k')}{2} \middle| k' \in X_{\mathrm{o}}\right].$$

Second, the bargaining surplus of a bank which participates exclusively in both markets, with capacity $k \in [k^{\star\star}, 1]$, is equal to

$$B(0, k, \mathrm{oc}) = |U_{gg}| \frac{m_{\mathrm{o}}}{m_{\mathrm{o}} + m_{\mathrm{oc}}} \left(\frac{1}{2} - \bar{g}_{\mathrm{o}}\right) \mathbb{E}\left[\frac{\max(k, k')}{2} \middle| k' \in X_{\mathrm{o}} \backslash X_{\mathrm{oc}}\right].$$

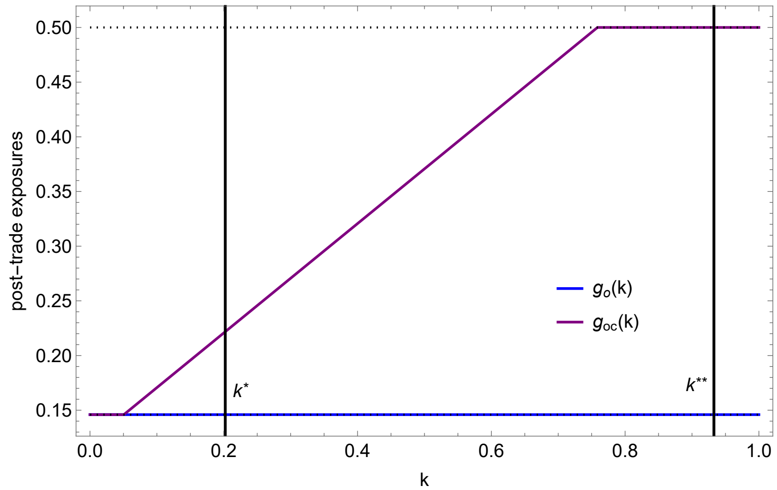

Figure 7: The post-trade exposures of a bank that participate exclusively in the OTC market (blue curve), or in both markets (purple curve), as functions of its capacity, k, when $k^\star = 0.202213,\ k^{\star\star} = 0.932682$.

Therefore, the average bargaining surplus computes as

$$\begin{aligned}\overline{B} &= \frac{m_{\rm o}}{m_{\rm o}+m_{\rm oc}}|U_{gg}|\left(\frac{1}{2}-\bar{g}_{\rm o}\right)\mathbb{E}\left[\frac{\max(k',k'')}{2}\middle| k'\in X_{\rm o}\backslash X_{\rm oc}, k''\in X_{\rm o}\right] \\ &+ \frac{m_{\rm oc}}{m_{\rm o}+m_{\rm oc}}|U_{gg}|\left(\frac{1}{2}-\bar{g}_{\rm o}\right)\frac{m_{\rm o}}{m_{\rm o}+m_{\rm oc}}\mathbb{E}\left[\frac{\max(k',k'')}{2}\middle| k'\in X_{\rm oc}, k''\in X_{\rm o}\backslash X_{\rm oc}\right] \\ &= |U_{gg}|\left(\frac{1}{2}-\bar{g}_{\rm o}\right)\frac{m_{\rm o}}{m_{\rm o}+m_{\rm oc}}2\bar{g}_{\rm o}\end{aligned}$$

With *Mathematica*, by plugging in $k^\star = 0.202213,\ k^{\star\star} = 0.932682$, and normalizing $|U_{gg}| = 1$, we find $\overline{B} = 0.0775387$. Next, we compute the bargaining surpluses of the marginal bank, that are

$$B(0,k^\star,{\rm o}) = |U_{gg}|\left(\frac{1}{2}-\bar{g}_{\rm o}\right)\left(\frac{m_{\rm o}}{m_{\rm o}+m_{\rm oc}}\frac{k^\star}{2}+\frac{m_{\rm oc}}{m_{\rm o}+m_{\rm oc}}\frac{1+k^{\star\star}}{4}\right)$$

and

$$B(0,k^{\star\star},{\rm oc}) = |U_{gg}|\left(\frac{1}{2}-\bar{g}_{\rm o}\right)\frac{m_{\rm o}}{m_{\rm o}+m_{\rm oc}}\frac{k^{\star\star}}{2}$$

With *Mathematica*, we find $B(0,k^\star,{\rm o}) = 0.0695795$ and $B(0,k^{\star\star},{\rm oc}) = 0.123867$ which implies that

$$B(0,k^\star,{\rm o}) < \overline{B} < B(0,k^{\star\star},{\rm oc}).$$

Therefore moving the bank $k^\star$ (exclusively) to the centralized market is welfare improving, while moving the bank $k^{\star\star}$ (exclusively) to the centralized market is welfare deteriorating.

A.11 Proof of Proposition 6

Heterogeneity in trading capacities. Since all $\omega = 0$-banks who participate in the OTC market have the same exposure, the bilateral trades between them are not uniquely determined. We make the natural assumption that, when two $\omega = 0$-traders meet, they "swap" the exposures their banks acquired from $\omega = 1$-banks, and vice versa when two $\omega = 1$-traders meet. Precisely, let

$$\bar{\gamma}(k) \equiv \mathbb{E}\left[\frac{k+k'}{2} \,\middle|\, \mathrm{o}\right] = \frac{k + \mathbb{E}\left[k' \,\middle|\, \mathrm{o}\right]}{2},$$

denote the net trade of a $(\omega = 0, k)$-bank with all $\omega = 1$-banks. We assume that, when an $(\omega = 0, k)$-trader meets an $(\omega = 0, k')$-trader, their bilateral trade is $\bar{\gamma}(k') - \bar{\gamma}(k) = \frac{k'-k}{2}$. It is easy to check that these bilateral exposures satisfy the trading capacity constraint. Moreover, when aggregated across all possible $\omega = 0$-counterparties, these swaps mechanically equalize exposure of all $\omega = 0$-banks who participate in the OTC market. Hence, these swaps implement the equilibrium post-trade exposure. Given our selection for bilateral trade, the net and gross volume are:

$$\begin{aligned}
NV(0,k,\mathrm{o}) &= g = \frac{\mathbb{E}\left[k' \,|\, \mathrm{o}\right]}{2} \\
GV(0,k,\mathrm{o}) &= \frac{1}{2}\frac{k + \mathbb{E}\left[k' \,|\, \mathrm{o}\right]}{2} + \frac{1}{2}\frac{\mathbb{E}\left[|k'-k| \,|\, \mathrm{o}\right]}{2}.
\end{aligned}$$

Heterogeneity in endowment. With heterogeneous endowments, the net and gross volume for a bank with endowment ω are:

$$\begin{aligned}
NV(\omega,k,\mathrm{o}) &= k\left(1 - 2N\left[\omega \,|\, \mathrm{o}\right]\right) \\
GV(\omega,k,\mathrm{o}) &= k.
\end{aligned}$$

Applying the Leibniz' rule to the formulas above whenever necessary, one obtains the (in)equalities stated in the Proposition.

Supplement to "A Theory of Participation in OTC and Centralized Markets"

This online appendix contains proofs omitted from the printed manuscript.

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B Other proofs

B.1 Omitted arguments from the proof of Proposition 2

In this section we proceed with weaker assumptions than in the proposition:

$$N(X_{\mathrm{o}}) + \varepsilon\, n(X_{\mathrm{o}}) > 0 \tag{30}$$

$$N(X_{\mathrm{c}}) + \varepsilon\, n(X_{\mathrm{c}}) > 0, \tag{31}$$

for all sufficiently small $\varepsilon > 0$. These two conditions hold in particular under the maintained assumption of the proposition, namely, if there is strictly positive participation in the market and if (n^+, n^-) is an admissible direction of reallocation. However they are more general: they allow us also calculate the marginal social value of creating some positive participation in an empty market: e.g. $N(X_{\mathrm{c}}) = 0$ and $n(X_{\mathrm{c}}) > 0$.

B.1.1 Right-hand differentiability

In this Section we show that:

Lemma 19. *Given any selection $(\gamma^\star, \varphi^\star)(\varepsilon)$ of the maximum correspondence:*

$$\frac{dW^\star}{d\varepsilon}(0^+) = \lim_{\varepsilon \to 0^+} \frac{\partial L}{\partial \varepsilon}((\gamma^\star, \varphi^\star)(\varepsilon), \varepsilon) \geq \max_{(\gamma^\star, \varphi^\star) \in \Lambda^\star(0)} \frac{\partial L}{\partial \varepsilon}((\gamma^\star, \varphi^\star), 0).$$

To that end, we check that the assumptions of Theorem 1, 3 and Corrolary 4 in Milgrom and Segal (2002) are satisfied in our setting. The main technical difficulty is to establish the result of Corrolary 4: the right-derivative can be calculated by taking the maximum of the partial derivative over all maximizers. This is a result that Milgrom and Segal (2002) provide in their Corollary 4 for continuous functions on compact choice sets. Since our choice set is only weakly compact, we must check that

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the required continuity properties hold in the weak topology. A useful preliminary result is to note that the functions g, $\partial g/\partial\varepsilon$, and $\partial^2 g/\partial\varepsilon^2$ are all uniformly bounded in $(x,\gamma,\varphi,\varepsilon)\in X\times\Lambda\times[0,\bar{\varepsilon}]$, for some $\bar{\varepsilon}$ small enough. For g, this follows directly because φ, γ, and ω are uniformly bounded. The first derivatives of g can be calculated explicitly as:

$$\frac{\partial g}{\partial\varepsilon}=\int\gamma(x,x')\,\frac{dn(x')N(X_{\mathrm{o}})-dN(x')n(X_{\mathrm{o}})}{\left[N(X_{\mathrm{o}})+\varepsilon\,n(X_{\mathrm{o}})\right]^2}. \tag{32}$$

The result then follows because γ is uniformly bounded, because N and n are finite signed measures, and from (30) and (31), which ensure that the numerators, $N(X_{\mathrm{o}})+\varepsilon n(X_{\mathrm{o}})$ and $N(X_{\mathrm{c}})+\varepsilon n(X_{\mathrm{c}})$ are bounded away from zero for small enough ε. Similar arguments imply uniform boundedness of the second derivative

$$\frac{\partial^2 g}{\partial\varepsilon^2}=-\,2n(X_{\mathrm{o}})\int\gamma(x,x')\,\frac{dn(x')N(X_{\mathrm{o}})-dN(x')n(X_{\mathrm{o}})}{\left[N(X_{\mathrm{o}})+\varepsilon\,n(X_{\mathrm{o}})\right]^3}.$$

Uniform boundedness allows us to apply Leibniz' rule to differentiate under the integral sign, and obtain the first and second partial derivative of the Lagrangian L with respect to ε:

$$\begin{aligned}\frac{\partial L}{\partial\varepsilon}=&\int U\left[g(x,\gamma,\varphi,\varepsilon)\right]\,dn(x)+\int U_g\left[g(x,\gamma,\varphi,\varepsilon)\right]\frac{\partial g}{\partial\varepsilon}(x,\gamma,\varphi,\varepsilon)\,\left[dN(x)+\varepsilon dn(x)\right]\\&-U_g\left(\frac{1}{2}\right)\int\varphi(x)\,dn(x)\\\frac{\partial^2 L}{\partial\varepsilon^2}=&2\int U_g\left[g(x,\gamma,\varphi,\varepsilon)\right]\frac{\partial g}{\partial\varepsilon}(x,\gamma,\varphi,\varepsilon)\,dn(x)\\&+\int U_{gg}\left[g(x,\gamma,\varphi,\varepsilon)\right]\left[\frac{\partial g}{\partial\varepsilon}(x,\gamma,\varphi,\varepsilon)\right]^2\,\left[dN(x)+\varepsilon dn(x)\right]\\&+\int U_g(x,\gamma,\varphi,\varepsilon)\frac{\partial^2 g}{\partial\varepsilon^2}(x,\gamma,\varphi,\varepsilon)\left[dN(x)+\varepsilon dn(x)\right].\end{aligned}$$

The uniform boundedness properties above imply that both $\partial L/\partial\varepsilon$ and $\partial^2 L/\partial\varepsilon^2$ are uniformly bounded in $(\gamma,\varphi,\varepsilon)$. This further implies that both L and $\partial L/\partial\varepsilon$ are Lipchitz continuous functions of ε, with Lipchitz coefficients that do not depend on γ or φ. Therefore, the equi-continuity and equi-differentiability properties required in Theorem 1 and 3 in Milgrom and Segal (2002) hold and Lemma 19 follows.

B.1.2 The right-hand derivative maximizes marginal social value

Next, we show that the inequality in Lemma 19 is, in fact, an equality:

Lemma 20. *The right-hand derivative maximizes marginal social value:*

$$\frac{dW^{\star}}{d\varepsilon}(0^{+})=\max_{(\gamma^{\star},\varphi^{\star})\in\Lambda^{\star}(0)}\frac{\partial L}{\partial\varepsilon}(\gamma^{\star},\varphi^{\star},0). \tag{33}$$

We adapt the argument of Corollary 4 in Milgrom and Segal. To that end consider a sequence $\varepsilon_m \to 0^+$ and some associated sequence of trades $(\gamma_m^\star, \varphi_m^\star) \in \Lambda^\star(\varepsilon_m)$. Let $g_m^\star(x) \equiv g(x, \gamma_m^\star, \varphi_m^\star, \varepsilon_m)$ and $\partial g_m^\star/\partial\varepsilon(x) \equiv \partial g/\partial\varepsilon(x, \gamma_m^\star, \varphi_m^\star, \varepsilon_m)$. Similarly, let $g^\star(x) \equiv g(x, \gamma^\star, \varphi^\star, 0)$ and $\partial g^\star/\partial\varepsilon = \partial g/\partial\varepsilon(x, \gamma^\star, \varphi^\star, 0)$.

Weak convergence. Given that that bilateral exposures are uniformly bounded, the Riez Weak Compactness Theorem (Royden and Fitzpatrick, 2010, Section 19.4) allows us to successively extract weakly convergent subsequences, so that we can assume without loss of generality that (γ_m, φ_m) converges weakly to some $(\gamma^\star, \varphi^\star)$ for $\gamma^\star$ and $\varphi^\star$ in $L^2(N\times N)$, $L^2(N\times n)$, $L^2(n\times N)$ and $L^2(n\times n)$, and that the sequences of real numbers $\int U\left[g_m^\star(x)\right]\, dN(x)$, $\int U\left[g_m^\star(x)\right]\, dn(x)$ and $\int dU/dg[g_m^\star(x)]\partial g_m^\star/\partial\varepsilon(x)\, dN(x)$ all converge. It then follows from direct calculations using the explicit formula for partial derivatives shown in the proof of Lemma 19 that $g_m^\star(x)$ and $\partial g_m^\star/\partial\varepsilon(x)$ converge to $g^\star(x)$ and $\partial g^\star/\partial\varepsilon(x)$ weakly in $L^2(N)$ and $L^2(n)$.

Strong convergence and asymptotic optimality of post-trade exposures. Given that $g \mapsto \int U\left[g(x)\right]\, dN(x)$ is strongly continuous and convex, it is weakly upper semi-continuous (see Corollary 2.2 in Eckland and Témam, 1987), which implies that:

$$\int U\left[g^\star(x)\right]\, dN(x) - U_g\left(\frac{1}{2}\right)\int \varphi^\star(x)\, dN(x) \\ \geq \lim_{m\to\infty}\int U\left[g_m^\star(x)\right]\, dN(x) - U_g\left(\frac{1}{2}\right)\int \varphi_m^\star(x)\, dN(x). \quad (34)$$

Given any $(\gamma, \varphi) \in \Lambda$, the optimality of $(\gamma_m^\star, \varphi_m^\star)$ given the distribution $N + \varepsilon_m n$ implies that:

$$\int U\left[g_m^\star(x)\right]\, \left[dN(x) + \varepsilon_m dn(x)\right] - U_g\left(\frac{1}{2}\right)\int \varphi_m^\star(x)\, \left[dN(x) + \varepsilon_m dn(x)\right]) \\ \geq \int U\left[g(x, \gamma, \varphi, \varepsilon_m)\right]\, \left[dN(x) + \varepsilon_m dn(x)\right] - U_g\left(\frac{1}{2}\right)\int \varphi(x)\, \left[dN(x) + \varepsilon_m dn(x)\right]. \quad (35)$$

It can be easily checked that, holding (γ, φ) fixed, $g(x, \gamma, \varphi, \varepsilon_m) \to g(x, \gamma, \varphi, 0)$ strongly in $L^2(N)$. Given that $g \mapsto \int U\left[g(x)\right]\, dN(x)$ is strongly continuous, we can go to the limit in the inequality (35) and, combining with (34), we obtain:

$$\int U\left[g^\star(x)\right]\, dN(x) - U_g\left(\frac{1}{2}\right)\int \varphi^\star(x)\, dN(x) \\ \geq \lim_{m\to\infty}\int U\left[g_m^\star(x)\right]\, dN(x) - U_g\left(\frac{1}{2}\right)\int \varphi_m^\star(x)\, dN(x) \\ \geq \int U\left[g(x, \gamma, 0)\right]\, dN(x) - U_g\left(\frac{1}{2}\right)\int \varphi(x)\, dN(x).$$

It follows that $(\gamma^\star, \varphi^\star)$ is an optimum for $\varepsilon = 0$, i.e. $(\gamma^\star, \varphi^\star) \in \Lambda^\star(0)$. Taking the supremum over $(\gamma, \varphi) \in \Lambda$ implies that

$$\lim_{m\to\infty} \int U\left[g_m^\star(x)\right] dN(x) - U_g\left(\frac{1}{2}\right)\int \varphi_m^\star(x)\, dN(x)$$
$$= \int U\left[g^\star(x)\right] dN(x) - U_g\left(\frac{1}{2}\right)\int \varphi^\star(x)\, dN(x).$$

Notice that, for all (γ, ϕ) maximizing the Lagrangian, we must have $\int \phi(x)\, dN(x) = 0$.[4] Taken together this implies that

$$\lim_{m\to\infty} \int U\left[g_m^\star(x)\right] dN(x) = \int U\left[g^\star(x)\right] dN(x).$$

Since $U[g]$ is quadratic and $g_m^\star \to g^\star$ weakly in $L^2(N)$, it follows that $\int \left[g_m^\star(x)\right]^2 dN(x) \to \int \left[g^\star(x)\right]^2 dN(x)$. Therefore $g_m^\star \to g^\star$ weakly in $L^2(N)$, and the $L^2(N)$ norm of g_m converges to that of $g^\star$. It thus follows that $g_m^\star \to g^\star$ strongly in $L^2(N)$.

The derivative maximizes marginal social value. With these results in mind, consider

$$\frac{\partial L}{\partial \varepsilon}(\gamma_m^\star, \varphi_m^\star, \varepsilon_m) = \int U\left[g_m^\star(x)\right] dn(x) + \int U_g\left[g_m^\star(x)\right] \frac{\partial g_m^\star}{\partial \varepsilon}\left[dN(x) + \varepsilon_m dn(x)\right]$$
$$- U_g\left(\frac{1}{2}\right)\int \varphi_m^\star(x)\, dn(x). \tag{36}$$

Using the weak upper semi continuity of $g \mapsto \int U\left[g(x)\right] dn(x)$ as above, we obtain that

$$\int U\left[g^\star(x)\right] dn(x) \geq \lim_{m\to\infty} U\left[g_m^\star(x)\right] dn(x). \tag{37}$$

Now recall that $dU/dg[g(x)]$ is linear, that $g_m^\star \to g^\star$ strongly in $L^2(N)$, that $\partial g_m^\star/\partial\varepsilon$ is uniformly bounded and converges weakly in $L^2(N)$ toward $\partial g^\star/\partial\varepsilon$. It thus follows that $dU/dg\left[g_m^\star\right]\partial g_m^\star/\partial\varepsilon$ converges weakly in $L^2(N)$ towards $dU/dg\left[g^\star\right]\partial g^\star/\partial\varepsilon$. Also recall that $\varphi_m^\star$ is uniformly bounded and converges weakly in $L^2(N \times N)$. Together with (37), this allows us to go to the limit as in (36) and

[4] Indeed, since the Lagrangian is strictly concave in g, it must be that the optimal post-trade exposures are equal, N almost everywhere, for all (γ, ϕ) maximizing the Lagragian. Letting (γ^e, φ^e) denote some equilibrium trades, this means that

$$g(x) = \omega(x) + \int \gamma(x, x')\, dN(x') + \varphi(x) = \omega(x) + \int \gamma^e(x, x')\, dN(x') + \varphi^e(x),$$

since we know from before that equilibrium trade solve the planner's problem and so maximize the Lagrangian as well. The result follow after integrating against $dN(x)$, keeping in mind that trades in the OTC market must be bilaterally feasible and so net out to zero.

obtain:

$$\frac{\partial L}{\partial \varepsilon}(\gamma^\star, \varphi^\star, 0) = \int U\left[g^\star(x)\right]\, dn(x) + \int U_g\left[g^\star(x)\right] \frac{\partial g^\star}{\partial \varepsilon} dN(x) \\ - U_g\left(\frac{1}{2}\right) \int \varphi^\star(x)\, dn(x) \geq \lim_{m\to\infty} \frac{\partial L}{\partial \varepsilon}(\gamma_m^\star, \varphi_m^\star, \varepsilon_m).$$

Combining with Lemma 19, the result follows.

B.1.3 Equilibrium exposures maximize the partial derivative

We now calculate the partial derivative:

$$\frac{\partial L}{\partial \varepsilon}(\gamma, \varphi, 0) = \int U\left[g(x)\right]\, dn(x) + \int U_g\left[g(x)\right] \frac{\partial g}{\partial \varepsilon}(x)\, dN(x) - U_g\left(\frac{1}{2}\right) \int \varphi(x)\, dn(x), \tag{38}$$

where, to simplify notations, we have let $g(x) \equiv g(x, \gamma, \varphi, 0)$ and $\partial g/\partial\varepsilon(x) \equiv \partial g/\partial\varepsilon\,(x, \gamma, \varphi, 0)$.

Next, we simpifly the second term of (38). If $N(X_o) > 0$, then the formula (32) implies that

$$\int U_g\left[g(x)\right] \frac{\partial g}{\partial \varepsilon}(x)\, dN(x) = \int\int U_g\left[g(x)\right] \gamma(x, x') \frac{dn(x')N(X_o) - dN(x')n(X_o)}{N(X_o)^2}\, dN(x)$$
$$= \int\int U_g\left[g(x')\right] \gamma(x', x) \frac{dN(x')}{N(X_o)}\, dn(x) - n(X_o) \int\int\ U_g\left[g(x)\right] \gamma(x, x') \frac{dN(x)\, dN(x')}{N(X_o)^2}$$
$$= -\int\int U_g\left[g(x')\right] \gamma(x, x') \frac{dN(x')}{N(X_o)}\, dn(x) - n(X_o) \int\int U_g\left[g(x)\right] \gamma(x, x') \frac{dN(x)\, dN(x')}{N(X_o)^2}$$

where the second-to-last equality follows by exchanging the name of variables, namely replacing x by x' in the first term, and the last equality follows by bilateral feasibility, i.e. $\gamma(x', x) = -\gamma(x, x')$. In addition, we have that:

$$-n(X_o) \int\int U_g\left[g(x)\right] \gamma(x, x') \frac{dN(x)\, dN(x')}{N(X_o)^2}$$
$$= -\frac{n(X_o)}{2} \int\int U_g\left[g(x)\right] \gamma(x, x') \frac{dN(x)\, dN(x')}{N(X_o)^2} - \frac{n(X_o)}{2} \int\int U_g\left[g(x')\right] \gamma(x', x) \frac{dN(x)\, dN(x')}{N(X_o)^2}$$
$$= -\frac{n(X_o)}{2} \int\int U_g\left[g(x)\right] \gamma(x, x') \frac{dN(x)\, dN(x')}{N(X_o)^2} + \frac{n(X_o)}{2} \int\int U_g\left[g(x')\right] \gamma(x, x') \frac{dN(x)\, dN(x')}{N(X_o)^2}$$
$$= -\frac{n(X_o)}{2} \left\{U_g\left[g(x)\right] - U_g\left[g(x')\right]\right\} \gamma(x, x') \frac{dN(x)\, dN(x')}{N(X_o)^2},$$

where: the first equality follows by breaking the integral into two identical halves and exchanging the name of variables in the second term, replacing x by x' in the second half; the second equality follows by bilateral feasibility $\gamma(x', x) = -\gamma(x, x')$; and the third equality by collecting terms. Taken together we obtain that, if $N(X_{\mathrm{o}}) > 0$,

$$\int U_g\left[g(x)\right]\frac{\partial g}{\partial \varepsilon}(x)\,dN(x) = -\int\int U_g\left[g(x')\right]\gamma(x,x')d\mu(x')\,dn(x) \\ - \frac{n(X_{\mathrm{o}})}{2}\int\int\left\{U_g\left[g(x')\right] - U_g\left[g(x'')\right]\right\}\gamma(x',x'')d\mu(x')d\mu(x'') \quad (39)$$

where $d\mu(x) = dN(x)/N(X_{\mathrm{o}})$. Substituting (39) into (38), we arrive at:

Lemma 21. *For any $(\gamma,\varphi) \in \Lambda$:*

$$\begin{aligned}\frac{\partial L}{\partial \varepsilon}(\gamma,\varphi,0) = &\int U\left[g(x,\gamma,\varphi,0)\right]\,dn(x) \\ &- \int U_g\left[g(x',\gamma,\varphi,0)\right]\gamma(x,x')d\mu(x')\,dn(x) \\ &- \frac{n(X_{\mathrm{o}})}{2}\int\int\left\{U_g\left[g(x',\gamma,\varphi,0)\right] - U_g\left[g(x'',\gamma,\varphi,0)\right]\right\}\gamma(x',x'')d\mu(x')d\mu(x'') \\ &- \int U_g\left(\frac{1}{2}\right)\varphi(x)\,dn(x),\end{aligned}$$

where

$$d\mu \equiv \frac{dN}{N(X_{\mathrm{o}})} \text{ if } N(X_{\mathrm{o}}) > 0 \text{ and } d\mu \equiv \frac{dn}{n(X_{\mathrm{o}})} \text{ otherwise.}$$

While our calculations so far have assumed that $N(X_{\mathrm{o}}) > 0$, one sees that the formula of the Lemma is valid when $N(X_{\mathrm{o}}) = 0$ as well. Indeed, in that case one sees from its definition that $g(x,\gamma,\varphi,\varepsilon)$ does not depend on ε, hence $\partial g/\partial\varepsilon = 0$ and $\int U_g\left[g(x)\right]\frac{\partial g}{\partial\varepsilon}(x)\,dN(x) = 0$ as well, so the second and third term of the formula should be equal to zero. One easily check that this is the case when setting $d\mu(x) = dn(x)/n(X_{\mathrm{o}})$.

Notice that, in the formula, the measure used to calculate average depend on whether there is, under N, positive participation in the OTC market. If $N(X_{\mathrm{o}}) > 0$, then the average is calculated based on the conditional distribution of incumbent in the OTC market. If $N(X_{\mathrm{o}}) = 0$, then the average is calculated based on the conditional distribution of entrant.

To proceed we assume that $N(X_{\mathrm{o}}) > 0$ and $N(X_{\mathrm{c}}) > 0$.[5] We show:

Lemma 22. *Equilibrium trades solve the problem of maximizing $\partial L/\partial\varepsilon$ with respect to $(\gamma^\star,\varphi^\star) \in \Lambda^\star(0)$.*

[5] While the proposition restricts attention to the case of strictly positive participation in both market, it can be extended to the case of $N(X_{\mathrm{o}}) = 0$ or $N(X_{\mathrm{c}}) = 0$, after redefining the equilibrium in an appropriate way. Consider for example participation patterns such that $N(X_{\mathrm{o}}) = 0$ and $N(X_{\mathrm{c}}) > 0$, with a perturbation such that $n(X_{\mathrm{o}}) > 0$. Then, one needs to define equilibrium post-trade exposures when there is a large group of investor, of size $N(X_{\mathrm{c}} \setminus X_{\mathrm{o}})$, participating in the centralized market, and an "infinitesimal" group of investors, with type distribution $n(x)/n(X_{\mathrm{o}})$, participating in the OTC and possibly simultaneously in the centralized market.

Consider any socially optimal $(\gamma, \varphi) \in \Lambda^\star(0)$. Because the first-order conditions hold almost everywhere according to $N(\,\cdot\,|\,\mathrm{o}) \times N(\,\cdot\,|\,\mathrm{o})$, it follows that the integrand of the last term in the formula of Lemma 21 is equal to

$$\left| U_g \left[g(x')\right] - U_g \left[g(x'')\right] \right| \Gamma(x', x'')$$

almost everywhere according to $N(\,\cdot\,|\,\mathrm{o}) \times N(\,\cdot\,|\,\mathrm{o})$. Therefore, the second to last term is constant and equal to

$$-n(X_\mathrm{o}) \frac{\bar{B}}{2},$$

for any socially optimal $(\gamma, \varphi) \in \Lambda^\star(0)$.

Now let (γ, φ) denote a collection of equilibrium trades with the associated post-trade exposures g, and let $(\hat{\gamma}, \hat{\varphi})$ denote any collection of socially optimal trades with the associated post-trade exposures $\hat{g}$. We calculate:

$$\begin{aligned}
& U\left[g(x)\right] - \int U_g \left[g(x')\right] \gamma(x, x') \, dN(x' \,|\, \mathrm{o}) - U_g \left(\frac{1}{2}\right) \varphi(x) \\
& \quad - U\left[\hat{g}(x)\right] + \int U_g \left[\hat{g}(x')\right] \hat{\gamma}(x, x') \, dN(x' \,|\, \mathrm{o}) + U_g \left(\frac{1}{2}\right) \hat{\varphi}(x) \\
\geq & U_g\left[g(x)\right]\left[g(x) - \hat{g}(x)\right] - U_g \left(\frac{1}{2}\right) \varphi(x) + U_g \left(\frac{1}{2}\right) \hat{\varphi}(x) \\
& \quad - \int U_g \left[g(x')\right] \left\{\gamma(x, x') - \hat{\gamma}(x, x')\right\} \, dN(x' \,|\, \mathrm{o}) \\
\geq & \left[U_g\left[g(x)\right] - U_g \left(\frac{1}{2}\right)\right] \left[\varphi(x) - \hat{\varphi}(x)\right] \\
& \quad + \int \left\{U_g\left[g(x)\right] - U_g \left[g(x')\right]\right\} \left\{\gamma(x, x') - \hat{\gamma}(x, x')\right\} \, dN(x' \,|\, \mathrm{o}),
\end{aligned}$$

where: the first inequality follows by concavity, and because optimality implies that $g(x') = \hat{g}(x')$ almost everywhere according to N; the second inequality follows using the explicit expression of $g(x)$ in terms of $\gamma(x, x')$.

Both the first and the second term of the last inequality is positive because in equilibrium, the first-order conditions hold everywhere. Hence we have shown that the integrand corresponding to the first three terms in Lemma 21 is greatest when evaluated at equilibrium bilateral exposures, and the result follows.

B.2 Proof of Lemma 6

Direct calculations show that:

$$\frac{d}{d\omega}S(\omega,k,\mathrm{o}) = |U_{gg}|\,(g_{\mathrm{o}}-\omega)\left(\frac{dg_{\mathrm{o}}}{d\omega}-1\right) + |U_{gg}|k\,[2N(\omega\,|\,\mathrm{o})-1]\,\frac{dg_{\mathrm{o}}}{d\omega}$$
$$\frac{d}{d\omega}B(\omega,k,\mathrm{o}) = |U_{gg}|k\,[2N(\omega\,|\,\mathrm{o})-1]\,\frac{dg_{\mathrm{o}}}{d\omega},$$

where g_{o} and $dg_{\mathrm{o}}/d\omega$ denote, respectively, the post-trade exposure and its right-derivative for an ω-bank who participates in the OTC market. Using that $g_{\mathrm{o}}-\omega = k\,[1-2N(\omega\,|\,\mathrm{o})]$, we obtain:

$$\frac{d}{d\omega}\mathrm{MPV}(\omega,k,\mathrm{o}) = -|U_{gg}|k\,[1-2N(\omega\,|\,\mathrm{o})]\left[\frac{1}{2}\frac{dg_{\mathrm{o}}}{d\omega}-1\right].$$

Now use that $\frac{d}{d\omega}\mathrm{MPV}(\omega,k,\mathrm{c}) = -|U_{gg}|\left[\frac{1}{2}-\omega-\left(\frac{1}{2}-g_{\mathrm{c}}\right)\frac{dg_{\mathrm{c}}}{d\omega}\right]$, and obtain:

$$\frac{d}{d\omega}\left[\mathrm{MPV}(\omega,k,\mathrm{c})-\mathrm{MPV}(\omega,k,\mathrm{o})\right]$$
$$= -|U_{gg}|\left\{\frac{1}{2}-\omega-\left(\frac{1}{2}-g_{\mathrm{c}}\right)\frac{dg_{\mathrm{c}}}{d\omega}+k\,[1-2N(\omega\,|\,\mathrm{o})]\left[\frac{1}{2}\frac{dg_{\mathrm{o}}}{d\omega}-1\right]\right\},$$

where g_{c} and $dg_{\mathrm{c}}/d\omega$ denote, respectively, the post-trade exposure and its right-derivative for an ω-bank who participates in the centralized market. For $\omega<\omega^\star$, then $g_{\mathrm{c}}=\omega+k$, $dg_{\mathrm{c}}/d\omega = dg_{\mathrm{o}}/d\omega = 1$, $N(\omega\,|\,\mathrm{o})=0$, so that:

$$\frac{d}{d\omega}\left[\mathrm{MPV}(\omega,k,\mathrm{c})-\mathrm{MPV}(\omega,k,\mathrm{o})\right] = -|U_{gg}|\left\{\frac{1}{2}-\omega-\left(\frac{1}{2}-\omega-k\right)-\frac{k}{2}\right\} = -|U_{gg}|\frac{k}{2}<0.$$

Next, for $\omega\in\left[\omega^\star,\frac{1}{2}-k\right]$, $g_{\mathrm{c}}=\omega+k$, $dg_{\mathrm{c}}/d\omega=1$, $N(\omega\,|\,\mathrm{o}) = \frac{\omega-\omega^\star}{1-2\omega^\star}$ and $\frac{dg_{\mathrm{o}}}{d\omega} = 1-\frac{2k}{1-2\omega^\star}$. Substituting and rearranging, we obtain after a few lines of algebra:

$$\frac{d}{d\omega}\left[\mathrm{MPV}(\omega,k,\mathrm{c})-\mathrm{MPV}(\omega,k,\mathrm{o})\right] = -|U_{gg}|\left\{-(1-2\omega)\left(\frac{1}{4}+\left(\frac{k}{1-2\omega^\star}\right)^2\right)+k\right\}<0$$

because $\frac{k}{1-2\omega^\star}<\frac{1}{2}$. Finally, for $\omega\in\left[\frac{1}{2}-k,\frac{1}{2}\right]$, $dg_{\mathrm{c}}/d\omega=0$, $N(\omega\,|\,\mathrm{o}) = \frac{\omega-\omega^\star}{1-2\omega^\star}$ and $\frac{dg_{\mathrm{o}}}{d\omega} = 1-\frac{2k}{1-2\omega^\star}$. Substituting and rearranging, we obtain after a few lines of algebra:

$$\frac{d}{d\omega}\left[\mathrm{MPV}(\omega,k,\mathrm{c})-\mathrm{MPV}(\omega,k,\mathrm{o})\right] = -|U_{gg}|\left\{\frac{1-2\omega}{1-2\omega^\star}\left(\frac{1}{2}-\omega^\star-k\right)\left(\frac{1}{2}+\frac{k}{1-2\omega^\star}\right)\right\}\le 0,$$

with equality if $\omega=\frac{1}{2}$.

Using the symmetry, i.e.,

$$\begin{aligned}
MPV(1-\omega, k, \mathrm{c}) &= MPV(\omega, k, \mathrm{c}) \\
MPV(1-\omega, k, \mathrm{o}) &= MPV(\omega, k, \mathrm{o}),
\end{aligned}$$

one easily sees that, for $\omega \in \left[\frac{1}{2}, 1\right]$

$$\frac{d}{d\omega}\left[\mathrm{MPV}(\omega, k, \mathrm{c}) - \mathrm{MPV}(\omega, k, \mathrm{o})\right] \geq 0,$$

with equality if $\omega = \frac{1}{2}$.

B.3 Proof of Lemma 7

Let w_0 denote the maximized value of the planner's problem and we consider the sets

$$\begin{aligned}
A &\equiv \left\{(w, y) \in \mathbb{R}^2 \,:\, W(\varphi, \gamma) \geq w \text{ and } \int \varphi(x)\, dN(x) = y \text{ for some } (\varphi, \gamma) \in \Lambda\right\} \\
B^+ &\equiv \left\{(w, y) \in \mathbb{R}^2 \,:\, w \geq w_0 \text{ and } y \geq 0\right\} \\
B^- &\equiv \left\{(w, y) \in \mathbb{R}^2 \,:\, w \geq w_0 \text{ and } y \leq 0\right\}.
\end{aligned}$$

It is clear that A, B^+ and B^- are all convex sets. We claim that either $A \cap \mathring{B}^+ = \emptyset$ or $A \cap \mathring{B}^- = \emptyset$. Otherwise, there would be some trades (φ^+, γ^+) and (φ^-, γ^-) such that

$$W(\varphi^+, \gamma^+) > w_0 \text{ and } \int \varphi^+(x)\, dN(x) > 0,$$

and

$$W(\varphi^-, \gamma^-) > w_0 \text{ and } \int \varphi^-(x)\, dN(x) < 0.$$

Clearly, if one forms the convex combination of (φ^+, γ^+) and (φ^-, γ^-) such that (40) holds, one obtains by concavity of W and convexity of Λ a feasible trade that attains a value larger than w_0, and we have reached a contradiction. Suppose for now without loss of generality that $A \cap \mathring{B}^- = \emptyset$. The separating hyperplane theorem then implies that there exists $(w^\star, y^\star) \in \mathbb{R}^2$, $(w^\star, y^\star) \neq (0, 0)$, such that:

$$w^\star\, w_A + y^\star\, y_A \leq w^\star\, w_B + y^\star\, y_B$$

for all $(w_A, y_A) \in A$ and $(w_B, y_B) \in B^-$. It is clear that $w^\star \geq 0$ otherwise the inequality would be violated for w_B large enough. If $w^\star = 0$ then using that $(w_0, 0) \in B^-$, we obtain that $y^\star\, y_A \leq 0$ for all $(w_A, y_A) \in A$. But since one can choose $(\varphi, \gamma) \in \Lambda$ such that $\int \varphi(x)\, dN(x) > 0$ or < 0, this implies that $y^\star = 0$, in contradiction with the fact that $(w^\star, y^\star) \neq (0, 0)$. Hence, $w^\star > 0$. Normalizing it to one without loss of generality, we obtain:

$$w^\star w_A + y^\star y_A \leq w_0,$$

for all $(w_A, y_A) \in A$. But this upper bound is achieves at the optimum, which implies the stated result with $g_{\rm c} \equiv U_g^{-1}(-y^\star)$.

B.4 Proof of Lemma 8

Define the following two functions. First:

$$\begin{aligned}\overline{V}(g) =&\omega(x) + \Phi(x)\mathbb{I}_{\{g \leq h_{\rm c}\}} - \Phi(x)\mathbb{I}_{\{g < h_{\rm c}\}} \\ &+ \int \Gamma(x, x')\mathbb{I}_{\{g \leq h(x')\}}\, dN(x' \,|\, {\rm o}) - \int \Gamma(x', x)\mathbb{I}_{\{g > h(x')\}}\, dN(x' \,|\, {\rm o}).\end{aligned}$$

The function $\overline{V}(g)$ represents the maximum post-trade exposure of the bank of type x, if all its traders take position anticipating that the post-trade will be g. One easily sees that $\overline{V}(g)$ is decreasing and left-continuous. Similarly, we let:

$$\begin{aligned}\underline{V}(g) =&\omega(x) + \Phi(x)\mathbb{I}_{\{g < h_{\rm c}\}} - \Phi(x)\mathbb{I}_{\{g \geq h_{\rm c}\}} \\ &+ \int \Gamma(x, x')\mathbb{I}_{\{g < h(x')\}}\, dN(x' \,|\, {\rm o}) - \int \Gamma(x', x)\mathbb{I}_{\{g \geq h(x')\}}\, dN(x' \,|\, {\rm o}).\end{aligned}$$

The function $\underline{V}(g)$ represents the minimum post-trade exposure of a bank, if all its traders take position anticipating that the post-trade will be g. One easily sees that $\underline{V}(g)$ is decreasing and right-continuous. Moreover:

$$\overline{V}(g+) = \underline{V}(g) \text{ and } \overline{V}(g) = \underline{V}(g-)$$

where $g+$ and $g-$ denote right- and left-limits. Finally, given that pre-trade exposures, $\omega(x)$, are bounded, and given that $\Gamma(x, x')$ is bounded over the support of N, there exists $a < b$ such that $\underline{V}(g) \in [a, b]$ and $\overline{V}(g) \in [a, b]$ for all g. We then have:

Step 1. We show that a post-trade exposure g solves (17)-(19) if and only if the fixed-point problem $g \in [\underline{V}(g), \overline{V}(g)]$. For the "only if" part, take a solution of (17)-(19) and use the optimality conditions (18) and (19) to show that it belongs to $[\underline{V}(g), \overline{V}(g)]$. For the "if" part, take some $g \in [\underline{V}(g), \overline{V}(g)]$, let

$$\begin{aligned}\varphi(x) &= \Phi(x)\mathbb{I}_{\{g < h_{\rm c}\}} + \left[(1-\alpha)\Phi(x) - \alpha\Phi(x)\right]\mathbb{I}_{\{g = h_{\rm c}\}} - \Phi(x)\mathbb{I}_{\{g > h_{\rm c}\}} \\ \gamma(x, x') &= \Gamma(x, x')\mathbb{I}_{\{g < h(x')\}} + \left[(1-\alpha)\Gamma(x, x') - \alpha\Gamma(x, x')\right]\mathbb{I}_{\{g = h(x')\}} - \Gamma(x', x)\mathbb{I}_{\{g > h(x')\}}\end{aligned}$$

where α is chosen such that:

$$g = (1 - \alpha)\overline{V}(g) + \alpha\underline{V}(g).$$

One can then directly verify that (φ, γ) solve the problem (17)-(19).

Step 2. Next we show that the fixed point problem $g \in [\underline{V}(g), \overline{V}(g)]$ has a unique solution. To show that a solution exists, we apply Kakutani's Fixed Point Theorem (see, e.g., Theorem 7 in Nachbar, 2017) to the correspondence

$$g \rightrightarrows V(g) \equiv [\underline{V}(g), \overline{V}(g)].$$

It is clear that $V(g)$ takes values that are convex sets included in $[a, b]$. To see that $V(g)$ has a closed graph consider any converging sequence $(g_n, v_n) \to (g, v)$ such that $v_n \in V(g_n)$ for all n. Then we can extract a subsequence of g_n such that either $g_n \leq g$ for all n or $g_n \geq g$ for all n. Suppose that we are in the former case (the latter case is symmetric). Then, since $\underline{V}(g)$ is decreasing, it follows that $v_n \geq \underline{V}(g_n) \geq \underline{V}(g)$. Going to the limit, we obtain $v \geq \underline{V}(g)$. Since $v_n \leq \overline{V}(g_n)$ and since $\overline{V}(g)$ is left-continuous, it follows that $v \leq \overline{V}(g)$. Therefore, $v \in V(g)$.

For uniqueness consider any $g \in V(g)$. Then $g \geq \underline{V}(g) = \overline{V}(g+)$. But since $\overline{V}(g)$ is decreasing, it follows that $g' > \overline{V}(g')$ for all $g' > g$. Hence, $g' \notin V(g')$. A similar argument applies to $g' < g$.

The second part of the Lemma follows directly because the proposed changes do not impact the functions $\underline{V}(g)$ and $\overline{V}(g)$, hence the fixed point remains the same.

B.5 Proof of Lemma 9

Step 1: modify trades in the centralized market. Let $U_g(g_c)$ denote the Lagrange multiplier on the constraint $\int \varphi(x)\, dN(x) = 0$ in the planner's problem. Then for all x such that (8) and (10) do not hold, we pick $\varphi(x)$ and $\gamma(x, x')$ that solve the partial equilibrium problem (17)-(19), given $h_c = g_c$ and $h(x') = g(x')$. By construction, these trades are both feasible and optimal for x (except perhaps for the bilateral feasibility constraint, which we address in the following paragraph). Moreover, because trades change for a measure zero set of x, they keep the post-trade exposures function the same N almost everywhere, they do not impact the post-trade exposures of any other x', nor do they impact the market-clearing condition in the centralized market.

Step 2: modify trades in the OTC market. First, set $\gamma(x, x') = 0$ for all $(x, x') \notin X_{\mathrm{o}}$. Define the following sets:

$$
\begin{aligned}
\Theta(x) &= \left\{x' \in X_{\mathrm{o}} \, \text{s.t.} \, (2) \text{ and } (8) \text{ hold for } (x, x')\right\} \\
\Theta &= \cup_{x \in X_{\mathrm{o}}} \{x\} \times \Theta(x) = \left\{(x, x') \in X_{\mathrm{o}}^2 \, \text{s.t.} \, (2) \text{ and } (8) \text{ hold for } (x, x')\right\} \\
\Psi &= \{x \in X_{\mathrm{o}} \, \text{s.t.} \, N(\Theta(x) \,|\, \mathrm{o}) = 1\} \, .
\end{aligned}
$$

One can show that, unsurprisingly, $N(\Psi \,|\, \mathrm{o}) = 1$. Indeed, since the first-order conditions hold almost everywhere:

$$
\begin{aligned}
N(\Theta \,|\, \mathrm{o}) = 1 \Leftrightarrow 1 &= \int_{x \in \Psi} N(\Theta(x) \,|\, \mathrm{o}) \, dN(x \,|\, \mathrm{o}) + \int_{x \notin \Psi} N(\Theta(x) \,|\, \mathrm{o}) \, dN(x \,|\, \mathrm{o}) \\
1 &= N(\Psi \,|\, \mathrm{o}) + \int_{X_{\mathrm{o}} \setminus \Psi} N(\Theta(x) \,|\, \mathrm{o}) \, dN(x \,|\, \mathrm{o}) \\
0 &= \int_{X_{\mathrm{o}} \setminus \Psi} \left[1 - N(\Theta(x) \,|\, \mathrm{o})\right] \, dN(x \,|\, \mathrm{o}),
\end{aligned}
$$

where the second line follows because $N(\Theta(x) \,|\, \mathrm{o}) = 1$ for all $x \in \Psi$, and the third line because $N(\Psi) = 1 - N(X_{\mathrm{o}} \setminus \Psi)$. But since $N(\Theta(x) \,|\, \mathrm{o}) < 1$ for all $x \notin \Psi$, the integrand in the last integral is strictly positive, so the only way that integral is zero is if $N(\Psi \,|\, \mathrm{o}) = 1$

Then, we define:

$$
A = \Theta \cap \Psi^2.
$$

The set A has measure one because it is the intersection of two sets of measure one. It contains pairs (x, x') with the following properties. First, since they belong to Θ, they together satisfy the feasibility condition (2) and the optimality condition (8). Second, since they belong to Ψ^2, they each satisfy the feasibility condition (2) and the optimality condition (8) with almost every other $\hat{x}$. Next we define:

$$
\begin{aligned}
B &= \left\{x \in X_{\mathrm{o}} \, : \, (x, x') \in A \text{ for some } x' \in X_{\mathrm{o}}\right\} \\
C &= X_{\mathrm{o}} \setminus B.
\end{aligned}
$$

Figure 8 illustrates these sets. The set B has also measure one because $A \subseteq B^2$ and, correspondingly, C has measure zero. Notice that for any $x \in B$, there exists (x, x') such that $(x, x') \in A$. But since $A \subseteq \Psi^2$, this implies that $x \in \Psi$ and so that $N(\Theta(x) \,|\, \mathrm{o}) = 1$. With these observations in mind, our modification goes as follows:

- For all $(x, x') \in A$, the feasibility condition (2) and the optimality condition (8) hold and so we keep $\gamma(x, x')$ the same.

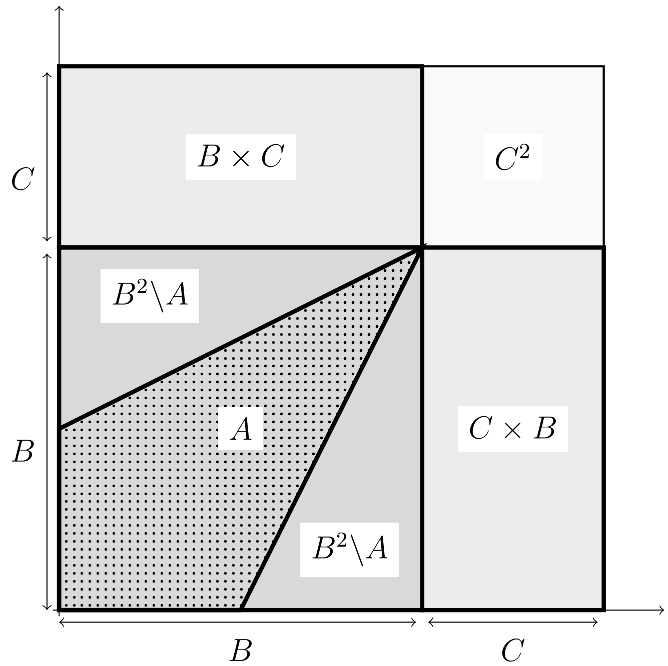

Figure 8: An illustration of the sets A, B, and C.

- For all $(x, x') \in B^2 \backslash A$, we modify $\gamma(x, x')$ so that it satisfies (2) and (8). Notice that since $N(\Theta(x) \mid \text{o}) = N(\Theta(x') \mid \text{o}) = 1$, these modifications concern a measure zero sets of counterparties for both x and x', and so they do not change the post-trade exposures $g(x)$ or $g(x')$.

- For all $(x, x') \in C \times B$, we pick $\gamma(x, x')$ and $g(x)$ that solves the fixed point problem of Section A.1.2. For any $x' \in B$, this changes the bilateral trades for a measure zero set of counterparties and so does not change $g(x')$.

- For $(x, x') \in C^2$, then we change the bilateral trades so that they satisfy (2) and (8). For either x or x', this changes the bilateral trades for a measure zero of counterparties, and so does not change $g(x)$ or $g(x')$.

These modification imply that the trades (φ, γ) now satisfy (2), (8), (10) everywhere in X^2, and

$$\int \varphi(x)\, dN(x) = 0. \tag{40}$$

The optimality conditions imply the capacity constraints (3) and (5). Thus, these trades are the basis of an equilibrium.

B.6 Proof of Lemma 10

Given that the third component of x, π, is discrete, we only need to prove that such a bound can be found conditional on π. Suppose for this proof that $\pi(x_1) = \pi(x_2) = \text{oc}$ (the other cases are similar) and that $g(x_1) < g(x_2)$. Consider first the trades in the centralized market. If $g(x_2) \leq g_c$, then $g(x_1) < g_c$, so (10) imply that $\varphi(x_1) = \Phi(x_1)$ and $\varphi(x_2) \leq \Phi(x_2)$. Hence, $\varphi(x_2) - \varphi(x_1) \leq \Phi(x_2) - \Phi(x_1)$. If $g(x_2) > g_c$, then $\varphi(x_2) = -\Phi(x_2)$ and $\varphi(x_1) \leq \Phi(x_1)$. Therefore, $\varphi(x_2) - \varphi(x_1) \leq \Phi(x_1) - \Phi(x_2)$. Hence, we obtain that, in both cases:

$$\varphi(x_2) - \varphi(x_1) \leq |\Phi(x_2) - \Phi(x_1)|.$$

Next, let us turn to the trades in the OTC market. The first-order condition for $\gamma(x, x')$ implies that, if $g(x') \leq g(x_1)$:

$$\gamma(x_1, x') \geq -\Gamma(x', x_1) \text{ and } \gamma(x_2, x') = -\Gamma(x', x_2).$$

On the other hand, if $g(x') > g(x_1)$:

$$\gamma(x_1, x') = \Gamma(x_1, x') \text{ and } \gamma(x_2, x') \leq \Gamma(x_2, x').$$

Therefore:

$$\begin{aligned}
&\int \gamma(x_2, x')\, dN(x' \,|\, \mathrm{o}) - \int \gamma(x_1, x')\, dN(x' \,|\, \mathrm{o}) \\
&\leq \int_{g(x') \leq g(x_1)} \left[\Gamma(x', x_1) - \Gamma(x', x_2)\right]\, dN(x' \,|\, X_\mathrm{o}) + \int_{g(x') > g(x_1)} \left[\Gamma(x_2, x') - \Gamma(x_1, x')\right]\, dN(x' \,|\, X_\mathrm{o}) \\
&\leq \sup_{x'} |\Gamma(x', x_2) - \Gamma(x', x_1)| + \sup_{x'} |\Gamma(x_2, x') - \Gamma(x_1, x')|.
\end{aligned}$$

Putting the upper bound for the difference in centralized and OTC market trades, and using that $g(x_2) - g(x_1) \geq 0$, we obtain:

$$\begin{aligned}
0 \leq g(x_2) - g(x_1) \leq \quad & |\omega(x_2) - \omega(x_1)| + |\Phi(x_2) - \Phi(x_1)| \\
& + \sup_{x'} |\Gamma(x', x_2) - \Gamma(x', x_1)| + \sup_{x'} |\Gamma(x_2, x') - \Gamma(x_1, x')|.
\end{aligned}$$

The inequality holds evidently if $g(x_1) = g(x_2)$ and symmetrically if $g(x_1) > g(x_2)$. This establishes the claim with $G(x_1, x_2)$ defined to be the function of the right-hand side inequality above.

B.7 Proof of Lemma 11

Proof that post-trade exposures are symmetric. Fix (ω, k, π) such that $\omega \leq \frac{1}{2}$. Consider some equilibrium collection of OTC trades $\gamma\left(\omega, k, \pi; \widetilde{\omega}, \widetilde{k}, \widetilde{\pi}\right)$ and centralized trades $\varphi\left(\omega, k, \pi\right)$ and the associated post-trade exposures $g\left(\omega, k, \pi\right)$. The alternative collection of OTC trades $\widehat{\gamma}\left(\omega, k, \pi; \widetilde{\omega}, \widetilde{k}, \widetilde{\pi}\right) = -\gamma\left(1-\omega, k, \pi; 1-\widetilde{\omega}, \widetilde{k}, \widetilde{\pi}\right)$ and the alternative collection of centralized trades $\widehat{\varphi}\left(\omega, k, \pi\right) = -\varphi\left(1-\omega, k, \pi\right)$

are feasible and generate post-trade exposures:

$$
\begin{aligned}
\widehat{g}(\omega,k,\pi) &= \omega+\widehat{\varphi}(\omega,k,\pi)+\int\widehat{\gamma}\left(\omega,k,\pi;\widetilde{\omega},\widetilde{k},\widetilde{\pi}\right)dN\left(\widetilde{\omega},\widetilde{k},\widetilde{\pi}\middle|\mathrm{o}\right)\\
&= \omega-\varphi(1-\omega,k,\pi)-\int\gamma\left(1-\omega,k,\pi;1-\widetilde{\omega},\widetilde{k},\widetilde{\pi}\right)dN\left(1-\widetilde{\omega},\widetilde{k},\widetilde{\pi}\middle|\mathrm{o}\right)\\
&= \omega-\varphi(1-\omega,k,\pi)-\int\gamma\left(1-\omega,k,\pi;\widetilde{\omega},\widetilde{k},\widetilde{\pi}\right)dN\left(\widetilde{\omega},\widetilde{k},\widetilde{\pi}\middle|\mathrm{o}\right)\\
&= 1-\left(1-\omega+\varphi(1-\omega,k,\pi)+\int\gamma\left(1-\omega,k,\pi;\widetilde{\omega},\widetilde{k},\widetilde{\pi}\right)dN\left(\widetilde{\omega},\widetilde{k},\widetilde{\pi}\middle|\mathrm{o}\right)\right)\\
&= 1-g(1-\omega,k,\pi).
\end{aligned}
$$

Now it is easy to see that $\widehat{\gamma}\left(\omega,k,\pi;\widetilde{\omega},\widetilde{k},\widetilde{\pi}\right)$ satisfies bilateral optimality since $\widehat{g}(\omega,k,\pi)<\widehat{g}\left(\widetilde{\omega},\widetilde{k},\widetilde{\pi}\right)$ is equivalent to $g\left(1-\widetilde{\omega},\widetilde{k},\widetilde{\pi}\right)<g(1-\omega,k,\pi)$. Similarly, the alternative centralized trades $\widehat{\varphi}(\omega,k,\pi)$ are optimal with our assumed market-clearing price $U_g\left(\frac{1}{2}\right)$, because $\widehat{g}(\omega,k,\pi)<\frac{1}{2}$ is equivalent to $\frac{1}{2}<g(1-\omega,k,\pi)$. Since equilibrium post-trade exposures are uniquely determined, we conclude from this that $\widehat{g}(\omega,k,\pi)=1-g(1-\omega,k,\pi)=g(\omega,k,\pi)$.

Proof that post-trade exposures of banks are increasing in endowment. Consider two banks with identical trading capacities and participation choices but different endowments, (ω,k,π) and (ω',k,π) with $\omega'>\omega$. Let $x=(\omega,k,\pi)$ be the type of the first bank, and $x'=(\omega',k,\pi)$ be the type of second bank. Assume, towards a contradiction, that $g(\omega,k,\pi)>g(\omega',k,\pi)$. Then, for all $y\in X_{\mathrm{o}}$ such that $g(y)\geq g(x)$, we have that:

$$
\begin{aligned}
g(x)\leq g(y) &\Rightarrow g(x')<g(y)\\
&\Rightarrow \gamma(x',y)=\mathbb{I}_{\{x'\in X_{\mathrm{o}}\}}\Gamma\left(k(x'),k(y)\right)=\mathbb{I}_{\{x\in X_{\mathrm{o}}\}}\Gamma\left(k(x),k(y)\right)\geq\gamma(x,y),
\end{aligned}
$$

where the first equality follows by optimality, the second equality follows by our maintained assumptions that $k(x')=k(x)$ and $\pi(x')=\pi(x)$, and the third inequality follows because of the trading capacity constraint. Likewise:

$$
g(x)\geq g(y)\Rightarrow\gamma(x,y)=-\mathbb{I}_{\{x\in X_{\mathrm{o}}\}}\Gamma\left(k(x),k(y)\right)=-\mathbb{I}_{\{x'\in X_{\mathrm{o}}\}}\Gamma\left(k(x'),k(y)\right)\leq\gamma(x',y),
$$

where the last inequality follows because $\gamma(x',y)$ is bounded by trading capacity. In all cases, $\gamma(x',y)\geq\gamma(x,y)$ for all $y\in X_{\mathrm{o}}$. The optimality of centralized market trades also implies $\varphi(x')\geq\varphi(x)$. Since $\omega(x')\geq\omega(x)$ by assumption, this implies that $g(x')\geq g(x)$, a contradiction.

Proof that $g(\omega,k,\pi) \leq 1/2$ for $\omega \leq 1/2$. By symmetry, $g(\omega,k,\pi) = 1 - g(1-\omega,k,\pi)$, and so, $g(\omega,k,\pi) + g(1-\omega,k,\pi) = 1$. Moreover, since we have just shown that post-trade exposures are increasing in endowment, we have $g(\omega,k,\pi) \leq g(1-\omega,k,\pi)$ and the result follows.

Proof that $1/2 \leq g(\omega,k,\pi)$ for $\omega > 1/2$. By symmetry, $g(\omega,k,\pi) = 1 - g(1-\omega,k,\pi)$. Since $1-\omega \leq 1/2$, we know from the previous paragraph that $g(1-\omega,k,\pi) \leq 1/2$ and the result follows.

Proof that post-trade exposures are weakly increasing in k for $\omega \leq \frac{1}{2}$. Let $\mu(\omega, k \,|\, \mathrm{o}) \equiv \int \left[\mathbb{I}_{\{\omega(x)\leq\omega \text{ and } k(x)\leq k \text{ and } \pi(x)\in\{\mathrm{o},\mathrm{oc}\}\}}\, dN(x)\right]/N(X_{\mathrm{o}})$ denote the fraction of OTC market traders with endowment less than ω and capacity less than k. Note that our assumptions $G\left(\omega \,|\, k\right) = 1 - G\left(1-\omega \,|\, k\right)$ and that participation patterns are symmetric in endowment imply $d\mu(\omega,k \,|\, \mathrm{o}) = d\mu(1-\omega, k \,|\, \mathrm{o})$. Now take any $k < k'$. If $g(\omega,k',\pi) = 1/2$, then since we have shown that $g(\omega,k,\pi)$ is bounded by $1/2$, it follows that $g(\omega,k',\pi) \geq g(\omega,k,\pi)$. If $g(\omega,k',\pi) < 1/2$, then we write:

$$
\begin{aligned}
&g(\omega,k',\pi) = \omega + \mathbb{I}_{\{\pi\in\{\mathrm{c},\mathrm{oc}\}\}}\Phi(k') \\
&+ \int \mathbb{I}_{\{\pi,\tilde{\pi}\in\{\mathrm{o},\mathrm{oc}\}\}}\mathbb{I}_{\{g(\tilde{\omega},\tilde{k},\tilde{\pi})<g(\omega,k',\pi)\}}\left[-\Gamma(k',\tilde{k}) + \Gamma(k',\tilde{k})\right]\, d\mu(\tilde{\omega},\tilde{k} \,|\, \mathrm{o}) \\
&+ \int \mathbb{I}_{\{\pi,\tilde{\pi}\in\{\mathrm{o},\mathrm{oc}\}\}}\mathbb{I}_{\{g(\tilde{\omega},\tilde{k},\tilde{\pi})=g(\omega,k',\pi)\}}\left[\gamma(\omega,k',\pi;\tilde{\omega},\tilde{k},\tilde{\pi}) + \Gamma(k',\tilde{k})\right]\, d\mu(\tilde{\omega},\tilde{k} \,|\, \mathrm{o}) \\
&+ \int \mathbb{I}_{\{\pi,\tilde{\pi}\in\{\mathrm{o},\mathrm{oc}\}\}}\mathbb{I}_{\{g(\tilde{\omega},\tilde{k},\tilde{\pi})>g(\omega,k',\pi)\}}\left[\Gamma(k',\tilde{k}) + \Gamma(k',\tilde{k})\right]\, d\mu(\tilde{\omega},\tilde{k} \,|\, \mathrm{o}).
\end{aligned}
$$

The second line considers all meetings with traders with endowment $\tilde{\omega}$ or $1-\tilde{\omega}$ and trading capacity $\tilde{k}$, who trade in the OTC market, and such that $g(\tilde{\omega},\tilde{k},\tilde{\pi}) < g(\omega,k',\pi)$. By symmetry, half of these traders have endowment $\tilde{\omega}$, and post trade exposure $g(\tilde{\omega},\tilde{k},\tilde{\pi}) < g(\omega,k',\pi)$, hence the bilateral trade is $-\Gamma(k',\tilde{k})$. The other half have endowment $1-\tilde{\omega}$ and post-trade exposure $g(1-\tilde{\omega},\tilde{k},\tilde{\pi}) = 1 - g(\tilde{\omega},\tilde{k},\tilde{\pi}) > 1/2$, hence the bilateral trade is $\Gamma(k',\tilde{k})$. The third and the fourth line have similar interpretations. Keeping in mind that $\gamma(\tilde{\omega},\tilde{k},\tilde{\pi};\omega,k',\pi) \geq -\Gamma(k',\tilde{k})$, we obtain that:

$$
\begin{aligned}
g(\omega,k',\pi) \geq\, &\omega + \mathbb{I}_{\{\pi\in\{\mathrm{c},\mathrm{oc}\}\}}\Phi(k') \\
&+ \int \mathbb{I}_{\{\pi,\tilde{\pi}\in\{\mathrm{o},\mathrm{oc}\}\}}\mathbb{I}_{\{g(\tilde{\omega},\tilde{k},\tilde{\pi})>g(\omega,k',\pi)\}}\Gamma(k',\tilde{k})\, d\mu(\tilde{\omega},\tilde{k} \,|\, \mathrm{o}).
\end{aligned} \tag{41}
$$

Now assume that for some $k < k'$ we have that $g(\omega, k, \pi) > g(\omega, k', \pi)$. We then obtain:

$$\begin{aligned}
&g(\omega, k, \pi) = \omega + \mathbb{I}_{\{\pi \in \{\mathrm{c,oc}\}\}} \Phi(k) \\
&+ \int \mathbb{I}_{\{\pi, \tilde{\pi} \in \{\mathrm{o,oc}\}\}} \mathbb{I}_{\{g(\tilde{\omega}, \tilde{k}, \tilde{\pi}) \leq g(\omega, k', \pi)\}} \left[-\Gamma(k, \tilde{k}) + \Gamma(k, \tilde{k}) \right] \, d\mu(\tilde{\omega}, \tilde{k} \,|\, \mathrm{o}) \\
&\quad + \int \mathbb{I}_{\{\pi, \tilde{\pi} \in \{\mathrm{o,oc}\}\}} \mathbb{I}_{\{g(\tilde{\omega}, \tilde{k}, \tilde{\pi}) > g(\omega, k', \pi)\}} \left[\gamma(\omega, k, \pi; \tilde{\omega}, \tilde{k}, \tilde{\pi}) + \gamma(\omega, k, \pi; 1 - \tilde{\omega}, \tilde{k}, \tilde{\pi}) \right] \, \mu(d\tilde{\omega}, d\tilde{k} \,|\, \mathrm{o}).
\end{aligned}$$

The second line considers all meetings with traders with endowment $\tilde{\omega}$ or $1 - \tilde{\omega}$ and trading capacity $\tilde{k}$, who trade in the OTC market, and such that $g(\tilde{\omega}, \tilde{k}, \tilde{\pi}) \leq g(\omega, k', \pi)$. By symmetry, half of these traders have endowment $\tilde{\omega}$, and post trade exposure $g(\tilde{\omega}, \tilde{k}, \tilde{\pi}) \leq g(\omega, k', \pi)$. But $g(\omega, k, \pi) > g(\omega, k', \pi)$, hence the bilateral trade is $-\Gamma(k, \tilde{k})$. The other half have endowment $1 - \tilde{\omega}$ and post-trade exposure $g(1 - \tilde{\omega}, \tilde{k}, \tilde{\pi}) = 1 - g(\tilde{\omega}, \tilde{k}, \tilde{\pi}) > 1/2$, hence the bilateral trade is $\Gamma(k, \tilde{k})$. The third line has a similar interpretation. Simplifying we obtain that:

$$\begin{aligned}
&g(\omega, k, \pi) = \omega + \mathbb{I}_{\{\pi \in \{\mathrm{c,oc}\}\}} \Phi(k) \\
&+ \int \mathbb{I}_{\{\pi, \tilde{\pi} \in \{\mathrm{o,oc}\}\}} \mathbb{I}_{\{g(\tilde{\omega}, \tilde{k}, \tilde{\pi}) > g(\omega, k', \pi)\}} \left[\gamma(\omega, k, \pi; \tilde{\omega}, \tilde{k}, \tilde{\pi}) + \gamma(\omega, k, \pi; 1 - \tilde{\omega}, \tilde{k}, \tilde{\pi}) \right] \, d\mu(\tilde{\omega}, \tilde{k} \,|\, \mathrm{o}).
\end{aligned} \tag{42}$$

Now recall that the trading capacity of k' is greater than that of k. Hence, in the centralized market, $\Phi(k) \leq \Phi(k')$. Moreover, in the OTC market, for any given counterparty, $\gamma(\omega, k, \pi; \tilde{\omega}, \tilde{k}, \tilde{\pi}) \leq \Gamma(k, \tilde{k}) \leq \Gamma(k', \tilde{k})$. Comparing (41) with (42), one then obtains that $g(\omega, k, \pi) \leq g(\omega, k', \pi)$, which is a contradiction.

Proof that post-trade exposures are weakly decreasing in k for $\omega > \frac{1}{2}$. By symmetry, $g(\omega, k, \pi) = 1 - g(1 - \omega, k, \pi)$. Then, the result follows from the earlier result that $g(1 - \omega, k, \pi)$ is increasing in k.

Proof that post-trade exposures are weakly increasing in centralized market participation for $\omega \leq \frac{1}{2}$. Pick some (ω, k) such that $\omega \leq 1/2$. If $g(\omega, k, \mathrm{oc}) = 1/2$ then we are done. If $g(\omega, k, \mathrm{oc}) < 1/2$ assume, towards a contradiction, that $g(\omega, k, \mathrm{o}) > g(\omega, k, \mathrm{oc})$. Then, for any counterparty in the OTC market, the optimality conditions imply that:

$$\gamma(\omega, k, \mathrm{o}; \tilde{\omega}, \tilde{k}, \tilde{\pi}) \leq \gamma(\omega, k, \mathrm{oc}; \tilde{\omega}, \tilde{k}, \tilde{\pi}).$$

Moreover, since $g(\omega, k, \mathrm{oc}) < 1/2$, it follows that $\varphi(\omega, k, \mathrm{oc}) = \Phi(k) \geq 0$. Sine the two bank start with the same endowment, and since the net trade of the oc bank is larger, we obtain that $g(\omega, k, \mathrm{oc}) \geq g(\omega, k, \mathrm{o})$, which is a contradiction.

Proof that post-trade exposures are weakly decreasing in centralized market participation for $\omega > \frac{1}{2}$. This follows by symmetry.

B.8 Proof of Lemma 12

First, let us rewrite the full appropriation surplus as follows

$$
\begin{aligned}
S(x) &= U\left[g(x)\right] - U\left[\omega(x)\right] - P_{\mathrm{c}}\,\varphi\left(x\right) - \int U_g\left[g(x')\right]\gamma\left(x,x'\right)dN(x'\,|\,\mathrm{o}) \\
&= U\left[g(x)\right] - U\left[\omega(x)\right] - P_{\mathrm{c}}\,\varphi\left(x\right) - \int U_g\left[g(x)\right]\gamma\left(x,x'\right)dN(x'\,|\,\mathrm{o}) \\
&+ \int \left(U_g\left[g(x)\right] - U_g\left[g(x')\right]\right)\gamma\left(x,x'\right)dN(x'\,|\,\mathrm{o}) \\
&= \overbrace{U\left[g(x)\right] - U\left[\omega(x)\right] - P_{\mathrm{c}}\,\varphi\left(x\right) - U_g\left[g(x)\right]\left(g(x) - \mathbb{I}_{\{x\in X_{\mathrm{c}}\}}\varphi\left(x\right) - \omega(x)\right)}^{=Q(x)} \\
&+ \underbrace{\int \left|U_g\left[g(x)\right] - U_g\left[g(x')\right]\right|\Gamma\left(x,x'\right)dN(x'\,|\,\mathrm{o})}_{=B(x)},
\end{aligned}
$$

where $B(x)$ is the bargaining surplus, and $Q(x)$ a new function labeled *quasi-competitive surplus*. Hence, the marginal private value can rewrite as

$$
\mathrm{MPV}(x) = Q(x) + \frac{B(x)}{2}.
$$

The quasi-competitive surplus can rewrite as follows

$$
\begin{aligned}
Q(x) &= U[g(x)] - U[\omega(x)] - U_g\left[\frac{1}{2}\right]\varphi(x) - U_g\left[g(x)\right]\left(g(x) - \varphi(x) - \omega(x)\right) \\
&= \frac{|U_{gg}|}{2}\left((g(x)-\omega(x))^2 + \varphi(x)(1-2g(x))\right) \\
\text{or alternatively,}\quad &= \frac{|U_{gg}|}{2}\left((g(x)-\varphi(x)-\omega(x))^2 + \varphi(x)(1-\varphi(x)-2\omega(x))\right)
\end{aligned}
$$

with $\varphi(x) = 0$ if the bank does not participate in the centralized market, and $g(x) = \varphi(x) + \omega(x)$ if the bank does not participate in the OTC market.

Second, define the net OTC trading volume of a bank $x = (\omega, k, \pi)$

$$
\overline{\gamma}\left(\omega,k,\pi\right) \equiv g\left(\omega,k,\pi\right) - \varphi\left(\omega,k,\pi\right) - \omega = \int \gamma\left(\omega,k,\pi;\widetilde{\omega},\widetilde{k},\widetilde{\pi}\right)dN\left(\widetilde{\omega},\widetilde{k},\widetilde{\pi}\middle|\mathrm{o}\right).
$$

Then, if $g(\omega,\pi,k) \neq 1/2$, the net OTC trading is decreasing in ω. Indeed, take $\omega < \omega'$ and fix some (k,π). If $g(\omega,k,\pi) < g(\omega',k,\pi)$, then the result is obvious because the ω bank will buy weakly more, from any counterparty in the OTC market, than the ω' bank. If $g(\omega,k,\pi) = g(\omega',k,\pi)$ then by our maintained assumption $g(\omega, k, \pi) \neq 1/2$ which implies that the centralized market trades of both banks are the same and equal to capacity. The results then follows from the definition of $\bar{\gamma}$.

Proof that the quasi-competitive surplus is V-shaped in ω. The symmetry of the problem makes obvious the symmetry of function $Q(x)$ with respect to $\omega = 1/2$. Indeed take $x = (\omega, k, \pi)$ and $x' = (1-\omega, k, \pi)$. Then $\varphi(x') = -\varphi(x)$, $g(x') = 1 - g(x)$ according to Proposition 1, and therefore $Q(x') = Q(x)$.

Then let us show that for $\omega < 1/2$, the quasi-competitive surplus weakly decreases with ω. If $g(x) = 1/2$, we have $Q(x) = |U_{gg}|\,(1/2-\omega)^2/2$ and as the post trade exposure remains flat when ω increases, the result follows. If $g(x) < 1/2$, then we necessarily have $\varphi(x) = \Phi(x)$ if the bank participates in the centralized, and 0 otherwise, which does not depend on ω. As shown above, we have that the net OTC trading volume, $g(x) - \varphi(x) - \omega$, decreases with ω, and that $1 - 2g(x)$ decreases with ω as $g(x)$ increases with ω. It implies that $Q(x)$ decreases.

Proof that the bargaining surplus is V-shaped in ω. For any $x = (\omega, k, \pi)$, let $y = (1-\omega, k, \pi)$, and for any x' similarly define y'. Keeping in mind that $U_g(g) - U_g(g') = |U_{gg}|\,(g' - g)$ we can write the bargaining surplus of y as:

$$
\begin{aligned}
B(y) &= |U_{gg}| \int \big|g(x') - g(y)\big| \Gamma(y, x')\, dN(x' \,|\, \mathrm{o}) = |U_{gg}| \int \big|1 - g(y') - g(y)\big| \Gamma(y, x')\, dN(x' \,|\, \mathrm{o}) \\
&= |U_{gg}| \int \big|g(y') - g(x)\big| \Gamma(y, x')\, dN(x' \,|\, \mathrm{o}).
\end{aligned}
$$

As Γ is a function of trading capacities only, we have $\Gamma(y, x') = \Gamma(x, x') = \Gamma(x, y')$. And thanks to symmetry, one can proceed to the change of variable, from x' to y', without changing the integral formula. This establishes that $B(y) = B(x)$, hence the bargaining surplus is symmetric.

Then, let us show that for $x_1 = (\omega_1, k, \pi)$ and $x_2 = (\omega_2, k, \pi)$ with $\omega_1 \leq \omega_2 \leq 1/2$, then $B(x_1) \geq B(x_2)$. One already knows that $g(x_1) \leq g(x_2)$. Denote $\lambda(x') = \Gamma(x_1, x') = \Gamma(x_2, x')$ Assuming that $\pi \in \{\mathrm{o}, \mathrm{oc}\}$, one can decompose the bargaining surplus as follows

$$
\begin{aligned}
B(x_1) = |U_{gg}| &\int \mathbb{I}_{\{g(x') \leq g(x_1)\}} \big(g(x_1) - g(x')\big)\, \lambda(x')\, dN(x' \,|\, \mathrm{o}) \\
+ |U_{gg}| &\int \mathbb{I}_{\{g(x_2) > g(x') > g(x_1)\}} \big(g(x') - g(x_1)\big)\, \lambda(x')\, dN(x' \,|\, \mathrm{o}) \\
+ |U_{gg}| &\int \mathbb{I}_{\{g(x') \geq g(x_2)\}} \big(g(x') - g(x_1)\big)\, \lambda(x')\, dN(x' \,|\, \mathrm{o})
\end{aligned}
$$

and

$$\begin{aligned}
B(x_2) = {} & |U_{gg}| \int \mathbb{I}_{\{g(x') \leq g(x_1)\}} \big(g(x_2) - g(x')\big)\, \lambda(x')\, dN(x' \,|\, \mathrm{o}) \\
& + |U_{gg}| \int \mathbb{I}_{\{g(x_2) > g(x') > g(x_1)\}} \big(g(x_2) - g(x')\big)\, \lambda(x')\, dN(x' \,|\, \mathrm{o}) \\
& + |U_{gg}| \int \mathbb{I}_{\{g(x') \geq g(x_2)\}} \big(g(x') - g(x_2)\big)\, \lambda(x')\, dN(x' \,|\, \mathrm{o})
\end{aligned}$$

Then

$$\begin{aligned}
B(x_1) - B(x_2) = {} & |U_{gg}| \int \mathbb{I}_{\{g(x') \leq g(x_1)\}} \left(g(x_1) - g(x_2)\right) \lambda(x')\, dN(x' \,|\, \mathrm{o}) \\
& + |U_{gg}| \int \mathbb{I}_{\{g(x_2) > g(x') > g(x_1)\}} \big(2g(x') - g(x_1) - g(x_2)\big)\, \lambda(x')\, dN(x' \,|\, \mathrm{o}) \\
& + |U_{gg}| \int \mathbb{I}_{\{g(x') \geq g(x_2)\}} \left(g(x_2) - g(x_1)\right) \lambda(x')\, dN(x' \,|\, \mathrm{o})
\end{aligned}$$

One further decomposes as

$$\begin{aligned}
B(x_1) - B(x_2) = {} & |U_{gg}| \int \mathbb{I}_{\{g(x') \leq g(x_1)\}} \left(g(x_1) - g(x_2)\right) \lambda(x')\, dN(x' \,|\, \mathrm{o}) \\
& + |U_{gg}| \int \mathbb{I}_{\{g(x_2) > g(x') > g(x_1)\}} \big(2g(x') - g(x_1) - g(x_2)\big)\, \lambda(x')\, dN(x' \,|\, \mathrm{o}) \\
& + |U_{gg}| \int \mathbb{I}_{\{1-g(x_2) \geq g(x') \geq g(x_2)\}} \left(g(x_2) - g(x_1)\right) \lambda(x')\, dN(x' \,|\, \mathrm{o}) \\
& + |U_{gg}| \int \mathbb{I}_{\{1-g(x_1) > g(x') > 1-g(x_2)\}} \left(g(x_2) - g(x_1)\right) \lambda(x')\, dN(x' \,|\, \mathrm{o}) \\
& + |U_{gg}| \int \mathbb{I}_{\{g(x') \geq 1-g(x_1))\}} \left(g(x_2) - g(x_1)\right) \lambda(x')\, dN(x' \,|\, \mathrm{o})
\end{aligned}$$

By a symmetry argument, one has

$$\begin{aligned}
& \int_{X_\mathrm{o}} \mathbb{I}_{\{1-g(x_1) > g(x') > 1-g(x_2)\}} \lambda(x')\, dN(x' \,|\, \mathrm{o}) = \int \mathbb{I}_{\{g(x_2) > g(x') > g(x_1)\}} \lambda(x')\, dN(x' \,|\, \mathrm{o}), \\
& \quad \text{and} \int \mathbb{I}_{\{g(x') \geq 1-g(x_1))\}} \lambda(x')\, dN(x' \,|\, \mathrm{o}) = \int \mathbb{I}_{\{g(x') \leq g(x_1)\}} \lambda(x')\, dN(x' \,|\, \mathrm{o})
\end{aligned}$$

Therefore, one obtains

$$
\begin{aligned}
B(x_1) - B(x_2) = {} & |U_{gg}| \int \mathbb{I}_{\{g(x') \leq g(x_1)\}} \left(g(x_1) - g(x_2)\right) \lambda(x')\, dN(x' \,|\, \mathrm{o}) \\
& + |U_{gg}| \int \mathbb{I}_{\{g(x_2) > g(x') > g(x_1)\}} \left(2g(x') - g(x_1) - g(x_2)\right) \lambda(x')\, dN(x' \,|\, \mathrm{o}) \\
& + |U_{gg}| \int \mathbb{I}_{\{1-g(x_2) \geq g(x') \geq g(x_2)\}} \left(g(x_2) - g(x_1)\right) \lambda(x')\, dN(x' \,|\, \mathrm{o}) \\
& + |U_{gg}| \int \mathbb{I}_{\{g(x_2) > g(x') > g(x_1)\}} \left(g(x_2) - g(x_1)\right) \lambda(x')\, dN(x' \,|\, \mathrm{o}) \\
& + |U_{gg}| \int \mathbb{I}_{\{g(x') \leq g(x_1)\}} \left(g(x_2) - g(x_1)\right) \lambda(x')\, dN(x' \,|\, \mathrm{o})
\end{aligned}
$$

and then

$$
\begin{aligned}
B(x_1) - B(x_2) = {} & |U_{gg}| \int \mathbb{I}_{\{g(x_2) > g(x') > g(x_1)\}} \left(2g(x') - 2g(x_1)\right) \lambda(x')\, dN(x' \,|\, \mathrm{o}) \\
& + |U_{gg}| \int \mathbb{I}_{\{1-g(x_2) \geq g(x') \geq g(x_2)\}} \left(g(x_2) - g(x_1)\right) \lambda(x')\, dN(x' \,|\, \mathrm{o}) \\
& \geq 0
\end{aligned}
$$

Proof that the MPV is increasing in k. Consider the case of a bank with $\omega < 1/2$. First, if the bank belongs to an atom so that $g(x) \leq 1/2$ is locally constant with k, then the MPV is obviously increasing. Indeed on the one hand the quasi-competitive surplus increases when the bank participates in the centralized market and buys more at a better price ($U_g\,[1/2]$ vs. $U_g\,[g(x)]$). This effect is captured by the term $\varphi(x)(1 - 2g(x))$ which increases (weakly). On the other hand, the bargaining surplus increases because the per-unit surplus is unchanged (as $g(x)$ is constant), and the gross trading volume increases with k.

Now, let us consider the case when the bank does not belong to an atom, that is $g(x) \leq 1/2$ is locally strictly increasing with k. Define the sets $\underline{X}(g) = \{x', g(x') < g \text{ or } g(x') > 1 - g\}$ and $\overline{X}(g) = \{x', g \leq g(x') \leq 1 - g\}$. First notice that if $x' = (\omega', k', \pi)$ belongs to one set, then the bank with symmetric endowment $\tilde{x}' \equiv (1 - \omega', k', \pi)$ belongs to the same set.

One can see the bank x engages in "pure intermediation" with banks in the set $\underline{X}(g(x))$, in the sense that its net trading volume is zero. Indeed, consider any counterparty x' such that $g(x') < g(x)$: then x sells to banks with $g(x')$ and buys an equal amount from banks with symmetric endowment, whose post-trade exposure is $1 - g(x')$. The combined surpluses obtained from trading with two banks with post-trade exposures of $g(x')$ and $1 - g(x')$ is $|U_{gg}|\,(1 - 2g(x'))\Gamma(x, x')$. The overall *intermediation* surplus is therefore

$$
|U_{gg}| \int_{\underline{X}(g(x))} \left(\frac{1}{2} - g(x')\right) \Gamma(x, x') dN(x' \,|\, \mathrm{o}).
$$

It is increasing in k as the size of the set $\underline{X}(g(x))$ increases, as well as $\Gamma(x, x')$.[6]

[6] The size of the set $\underline{X}(g(x))$ is increasing in k because $g(x)$ is increasing in k as implied by Proposition 1.

One can see the bank x does "pure directional trading" with the banks in the set $\overline{X}(g(x))$, that is x only buys from the banks in the set $\overline{X}(g)$, and so, its net trading volume corresponds to its OTC post-trade exposure, $g(x) - \varphi(x) - \omega$. For $g(x') \geq g(x)$, the combined surplus per-unit obtained from trading with a bank with $g(x')$ and a bank with $1 - g(x')$ is $|U_{gg}|\,(1 - 2g(x))$ which does not depend on x'. The overall *directional* surplus is therefore

$$|U_{gg}| \left(\frac{1}{2} - g(x)\right)(g(x) - \varphi(x) - \omega).$$

If one adds half this directional surplus with the quasi-competitive surplus, one obtains

$$\begin{aligned}
&\frac{|U_{gg}|}{2}\left((g(x)-\omega)^2 + \varphi(x)(1-2g(x)) + \left(\frac{1}{2} - g(x)\right)(g(x) - \varphi(x) - \omega)\right) \\
=&\frac{|U_{gg}|}{2}\left((g(x)-\omega)^2 + \left(\frac{1}{2} - g(x)\right)(g(x) + \varphi(x) - \omega)\right) \\
=&\frac{|U_{gg}|}{2}\left(\left(\frac{1}{2}-\omega\right)^2 - \left(\frac{1}{2} - g(x)\right)\left(\frac{1}{2} + g(x) - 2\omega\right) + \left(\frac{1}{2} - g(x)\right)(g(x) + \varphi(x) - \omega)\right) \\
=&\frac{|U_{gg}|}{2}\left(\left(\frac{1}{2}-\omega\right)^2 - \left(\frac{1}{2} - g(x)\right)\left(\frac{1}{2} - \varphi(x) - \omega\right)\right)
\end{aligned}$$

It is increasing with respect to the terms $g(x)$ and $\varphi(x)$. Therefore it is increasing in k.

B.9 Proof of Lemma 13

Post-trade exposures for π = o. We can write the post-trade exposure of a $\pi = 0$ bank with capacity k as:

$$g_{\mathrm{o}}(k) = \frac{m_{\mathrm{o}}}{m_{\mathrm{o}} + m_{\mathrm{oc}}}\left(\frac{1}{2}\int \gamma(k,k') f(k' \,|\, \mathrm{o})\, dk' + \frac{1}{2}\frac{k + \mathbb{E}\left[k' \,|\, \mathrm{o}\right]}{2}\right) + \frac{m_{\mathrm{oc}}}{m_{\mathrm{o}} + m_{\mathrm{oc}}}\frac{k + \mathbb{E}\left[k' \,|\, \mathrm{oc}\right]}{2}, \quad (43)$$

where $m_{\mathrm{o}} = F(k^\star)$ and $m_{\mathrm{oc}} = 1 - F(k^\star)$ denote the measure of $\pi = \mathrm{o}$ and $\pi = \mathrm{oc}$ in the OTC market. Taking expectations with respect to $f(k \,|\, \mathrm{o})$, we obtain the formula (28) in the Lemma.

To check that these post-trade exposures are indeed part of an equilibrium, we need to find bilaterally feasible and optimal trades that attain them. To do so we guess that:

$$\gamma(k,k') = \alpha \frac{k' - k}{2}.$$

Plugging this into equation (43), we find that post-trade exposures are constant and equal (28) if and only if:

$$\alpha = 1 + 2\frac{m_{\mathrm{oc}}}{m_{\mathrm{o}}}.$$

This bilateral trades are optimal because all $\pi = \mathrm{o}$ have the same exposure, so they are indifferent about any trade. They are bilaterally feasible because $\gamma(k,k') + \gamma(k',k) = 0$ by construction, and as long as:

$$\alpha \frac{k'-k}{2} \leq \frac{k'+k}{2},$$

which holds for all $(k,k') \in [\underline{k},\overline{k}]$ if and only if it holds for $k' = \bar{k}$ and $k = \underline{k}$. This can be simplifed to:

$$\frac{m_{\mathrm{oc}}}{m_{\mathrm{o}} + m_{\mathrm{oc}}} \overline{k} \leq \underline{k} \Leftrightarrow [1 - F(k^\star)]\, \overline{k} \leq \underline{k}.$$

Given our maintained assumption that $\underline{k} > 0$, this condition is always satisfied as long as $k^\star$ is close enough to $\overline{k}$.

Post-trade exposures of $\pi = \mathbf{oc}$, $k \in [k^\star, \overline{k}]$. We guess and verify that the atom property holds for these banks: as long as $k^\star$ is close enough to $\overline{k}$, these banks must have the same post-trade exposure.

We first argue that, for $[k^\star, \bar{k}]$, $g_{\mathrm{oc}}(k) > \bar{g}_{\mathrm{o}}$. Suppose, towards a contradiction, that $g_{\mathrm{oc}}(k) \leq g_{\mathrm{o}}$. Then it follows that

$$g_{\mathrm{oc}}(k) \geq \frac{k+K}{2}.$$

Indeed, the bank will trade to full capacity in the centralized market. Moreover, its net trade in the OTC market must, on net, be positive: indeed, it will buy strictly more from $\omega = 1$, who have strictly larger exposure, than it will sell to $\omega = 0$. But given that $\bar{k} + K > \mathbb{E}[k]$, we have reached a contradiction as long as $k^\star$ is close enough to $\bar{k}$.

Next, we argue that the post trade exposures must be constant. We already know from earlier work that the post-trade exposures must be weakly increasing in k and less than $1/2$. Suppose, towards a contradiction, that $g_{\mathrm{oc}}(k^\star \,|\, k^\star) < g_{\mathrm{oc}}(\overline{k} \,|\, k^\star)$. It follows that there must be some point $\hat{k}$ such that it is strictly increasing to the right. Moreover, since the function is continuous, it can be shown that there is some sequence $k_n \downarrow \hat{k}$, $k_n \neq \hat{k}$, such that the function is strictly increasing at each k_n.[7] At each k_n, post-trade exposures must be equal to:

$$g_{\mathrm{oc}}(k \,|\, k^\star) = \frac{k+K}{2} + \frac{m_{\mathrm{o}}}{m_{\mathrm{o}} + m_{\mathrm{oc}}} \times 0$$
$$+ \frac{m_{\mathrm{oc}}}{m_{\mathrm{o}} + m_{\mathrm{oc}}} \frac{1}{2} \left(- \int_{k^\star}^{k} \frac{k+k'}{2} f(k' \,|\, \mathrm{oc})\, dk' + \int_{k}^{\overline{k}} \frac{k+k'}{2} f(k' \,|\, \mathrm{oc})\, dk' + \int_{k^\star}^{\overline{k}} \frac{k+k'}{2} f(k' \,|\, \mathrm{oc})\, dk' \right)$$

[7] Formally one first note that the function has countably many "flat spots", defined as some interval $[k_1, k_2]$, $k_1 < k_2$ such that $g(k)$ is constant over the entire interval: indeed one can associate each flat spot to a distinct rational number." The result intuitively follows because the function must move continuously from one flat spot to the next.

The first term is the bank's trade in the centralized market. The second term is the net with $\pi = \mathrm{o}$ banks. This term is zero because the bank sells to $\pi = \mathrm{o}$, $\omega = 0$ banks, and an buys an equal amount from $\pi = 0$, $\omega = 1$ banks. The last term, on the second line, is the net trade with $\pi = \mathrm{oc}$ banks. Simplifying we obtain:

$$\begin{aligned} g_{\mathrm{oc}}(k) &= \frac{k+K}{2} + \frac{m_{\mathrm{oc}}}{m_{\mathrm{o}} + m_{\mathrm{oc}}} \int_{k}^{\overline{k}} \frac{k+k'}{2} f(k' \,|\, \mathrm{oc})\, dk' \\ &= \frac{k+K}{2} + \int_{k}^{\overline{k}} \frac{k+k'}{2} f(k')\, dk' \equiv \hat{g}_{\mathrm{oc}}(k) \end{aligned}$$

where we used that $m_{\mathrm{oc}}/m_{\mathrm{o}} + m_{\mathrm{oc}} = 1 - F(k^\star)$, and $f(k' \,|\, \mathrm{oc}) = f(k')/\left[1 - F(k^\star)\right]$. Hence, at each k_n, post-trade exposures are given by a function $\hat{g}_{\mathrm{oc}}(k)$ that is independent of $k^\star$. Taking derivatives, we obtain:

$$\hat{g}'_{\mathrm{oc}}(k) = \frac{1}{2} - k f(k) + \frac{1}{2}\left(1 - F(k)\right).$$

But if $\overline{k} f(\overline{k}) > 1/2$, then as long as $k^\star$ is close enough to $\overline{k}$, $\hat{g}'_{\mathrm{oc}}(k) < 0$ over $[k^\star, \overline{k}]$, which implies that $g_{\mathrm{oc}}(k_n \,|\, k^\star) > g_{\mathrm{oc}}(k_{n+1} \,|\, k^\star)$, which contradicts our earlier result that $g_{\mathrm{oc}}(k \,|\, k^\star)$ is weakly increasing.

Finally, we calculate the post-trade exposure as usual, by integrating the post-trade exposure formula across k. We find:

$$\begin{aligned} \bar{g}_{\mathrm{oc}} &= \frac{\mathbb{E}\left[k' \,|\, \mathrm{oc}\right] + K}{2} + \frac{m_{\mathrm{oc}}}{m_{\mathrm{o}} + m_{\mathrm{oc}}} \frac{\mathbb{E}\left[k' \,|\, \mathrm{oc}\right]}{2} \\ &= \frac{\mathbb{E}\left[k' \,|\, \mathrm{oc}\right] + K}{2} + \frac{1}{2}\int_{k^\star}^{\overline{k}} k' f(k')\, dk' = \frac{\bar{k} + K}{2} + o(1), \end{aligned}$$

as $k^\star \to \overline{k}$.

Post-trade exposures of $\pi = \mathrm{oc}$, $k \in [\underline{k}, k^\star]$. We argue that post-trade exposures must be given by the formula (29) in the Lemma. To see this, note first that $g_{\mathrm{oc}}(k) \geq \bar{g}_{\mathrm{o}}$. Otherwise, the net trade of a $\pi = \mathrm{oc}$ would be strictly larger than that of a $\pi = \mathrm{o}$ bank with the same k, a contradiction.

Note as well that, since $g_{\mathrm{oc}}(k)$ is weakly increasing and since $g_{\mathrm{oc}}(\overline{k}) = g_{\mathrm{oc}}$, it follows that $g_{\mathrm{oc}}(k) \leq g_{\mathrm{oc}}$ for all k.

If $g_{\mathrm{oc}}(k) = \bar{g}_{\mathrm{o}}$, then the net trade with $\pi = \mathrm{o}$ banks must be positive: indeed, the bank will buy to full capacity from $\pi = \mathrm{o}$, $\omega = 1$ bank, and sell less from $\pi = \mathrm{o}$, $\omega = 0$ banks. Hence, the net trade must be larger than $\frac{k+K}{2} + \frac{m_{\mathrm{oc}}}{m_{\mathrm{o}} + m_{\mathrm{oc}}} \frac{k + \mathbb{E}[k' \,|\, \mathrm{oc}]}{2}$, that is, larger than the net trade obtained by buying in the centralized market, buying from all $\pi = \mathrm{oc}$ banks, and not trading with $\pi = \mathrm{o}$ banks. Hence, formula (29) holds.

If $g_{\mathrm{oc}}(k) > g_{\mathrm{o}}$ and $g_{\mathrm{oc}}(k) < g_{\mathrm{oc}}$, then one sees by direct calculation that $g_{\mathrm{oc}}(k) = \frac{k+K}{2} + \frac{m_{\mathrm{oc}}}{m_{\mathrm{o}} + m_{\mathrm{oc}}} \frac{k + \mathbb{E}[k' \,|\, \mathrm{oc}]}{2}$, and formula (29) holds.

Finally, if $g_{\mathrm{oc}}(k) = \bar{g}_{\mathrm{oc}}$, then the net trades with $\pi = \mathrm{o}$ bank are zero, implying that the total net trade are less than the one obtained if buying from all $\pi = \mathrm{oc}$ banks, i.e. less than $\frac{k+K}{2} + \frac{m_{\mathrm{oc}}}{m_{\mathrm{o}}+m_{\mathrm{oc}}} \frac{k+\mathbb{E}[k' \,|\, \mathrm{oc}]}{2}$, and so formula (29) holds.

B.10 Proof of Lemma 14

The result is obvious for $\bar{g}_{\mathrm{o}}$. For $\bar{g}_{\mathrm{oc}}(k)$, note that it can be written:

$$g_{\mathrm{oc}}(k) = \max\left\{\bar{g}_{\mathrm{o}}^{\dagger} + o_1(1), \min\left\{\frac{\overline{k}+K}{2} + o_2(1), \frac{k+K}{2} + o_3(1)\right\}\right\},$$

because $m_{\mathrm{oc}} \to 0$ and $\mathbb{E}[k \,|\, \mathrm{oc}] \to \overline{k}$ as $k^{\star} \to \overline{k}$. Letting $\overline{o}(1) = \max\{o_1(1), o_2(1), o_3(1)\}$ and $\underline{o}(1) = \min\{o_1(1), o_2(1), o_3(1)\}$, we obtain that, for $k \in [\underline{k}, \overline{k}]$:

$$g_{\mathrm{oc}}^{\dagger}(k) + \underline{o}(1) \leq g_{\mathrm{oc}}(k) \leq g_{\mathrm{oc}}^{\dagger}(k) + \overline{o}(1),$$

and the result follows.

B.11 Proof of Lemma 15

MPV of $\pi = \mathrm{o}$. The marginal private value is calculated as follows:

$$\begin{aligned}
\mathrm{MPV}(k \,|\, \mathrm{o}) =& U(\bar{g}_{\mathrm{o}}) - U(0) - U_g(\bar{g}_{\mathrm{o}})\bar{g}_{\mathrm{o}} \\
&+ \frac{1}{2}\frac{m_{\mathrm{o}}}{m_{\mathrm{o}}+m_{\mathrm{oc}}}\frac{|U_{gg}|}{2}\left(1-\bar{g}_{\mathrm{o}}-\bar{g}_{\mathrm{o}}\right)\frac{k+\mathbb{E}[k' \,|\, \mathrm{o}]}{2} \\
&+ \frac{1}{2}\frac{m_{\mathrm{oc}}}{m_{\mathrm{o}}+m_{\mathrm{oc}}}\frac{|U_{gg}|}{2}\left(\bar{g}_{\mathrm{oc}}-\bar{g}_{\mathrm{o}}\right)\frac{k+\mathbb{E}[k' \,|\, \mathrm{oc}]}{2} \\
&+ \frac{1}{2}\frac{m_{\mathrm{oc}}}{m_{\mathrm{o}}+m_{\mathrm{oc}}}\frac{|U_{gg}|}{2}\left(1-\bar{g}_{\mathrm{oc}}-\bar{g}_{\mathrm{o}}\right)\frac{k+\mathbb{E}[k' \,|\, \mathrm{oc}]}{2}.
\end{aligned}$$

The first line is the fundamental surplus. The second, third, and fourth lines are the frictional surplus with, respectively, $\pi = \mathrm{o}$ and $\omega = 1$ banks, $\pi = \mathrm{oc}$ and $\omega = 0$ banks, $\pi = \mathrm{oc}$ and $\omega = 1$ banks. Keeping in mind that the fundamental surplus is $|U_{gg}|/2g^2$ and collecting terms, we obtain the formula shown in the Lemma.

MPV of $\pi = \mathbf{oc}$. The MPV is:

$$\begin{aligned}
\text{MPV}(k \,|\, \text{oc}) =& U(g_{\text{oc}}(k)) - U(0) - U_g(g_{\text{oc}}(k))g_{\text{oc}}(k) \\
&+ |U_{gg}| \left(\frac{1}{2} - g_{\text{oc}}(k)\right) \frac{k+K}{2} \\
&+ \frac{1}{2}\frac{m_{\text{o}}}{m_{\text{o}} + m_{\text{oc}}} \frac{|U_{gg}|}{2} \left(g_{\text{oc}}(k) - \bar{g}_{\text{o}}\right) \frac{k + \mathbb{E}\left[k' \,|\, \text{o}\right]}{2} \\
&+ \frac{1}{2}\frac{m_{\text{o}}}{m_{\text{o}} + m_{\text{oc}}} \frac{|U_{gg}|}{2} \left(1 - \bar{g}_{\text{o}} - g_{\text{oc}}(k)\right) \frac{k + \mathbb{E}\left[k' \,|\, \text{o}\right]}{2} \\
&+ \frac{1}{2}\frac{m_{\text{oc}}}{m_{\text{o}} + m_{\text{oc}}} \frac{|U_{gg}|}{2} \left(\bar{g}_{\text{oc}} - g_{\text{oc}}(k)\right) \frac{k + \mathbb{E}\left[k' \,|\, \text{oc}\right]}{2} \\
&+ \frac{1}{2}\frac{m_{\text{oc}}}{m_{\text{o}} + m_{\text{oc}}} \frac{|U_{gg}|}{2} \left(1 - \bar{g}_{\text{oc}} - g_{\text{oc}}(k)\right) \frac{k + \mathbb{E}\left[k' \,|\, \text{oc}\right]}{2}.
\end{aligned}$$

The first line is the fundamental surplus. The second line is the surplus (which as Semih pointed out should not be called frictional) that banks earn because, in the centralized market, they buy at price $U_g(1/2)$ and not at their marginal value. The third, fourth, fifth and sixth lines are the frictional surplus with, respectively, $\pi = \text{o}$ and $\omega = 0$ banks, $\pi = \text{o}$ and $\omega = 1$ banks, $\pi = \text{oc}$ and $\omega = 0$ banks, $\pi = \text{oc}$ and $\omega = 1$ banks. For these terms, keep in mind that $\bar{g}_{\text{o}} \leq g_{\text{oc}}(k) \leq \bar{g}_{\text{oc}}$. Simplifying, we obtain the formula in the Lemma.

B.12 Proof of Lemma 16

The result is obvious for the MPV. It is also clear that

$$\frac{d\text{MPV}}{dk}(k \,|\, \text{o}) = \frac{d\text{MPV}^{\dagger}}{dk}(k \,|\, \text{o}) + o(1),$$

so we only have to make sure that the result holds for $\text{MPV}(k \,|\, \text{oc})$. We have:

$$\begin{aligned}
\frac{d\text{MPV}}{dk}(k \,|\, \text{oc}) =& |U_{gg}| \frac{dg_{\text{oc}}}{dk} \left(g_{\text{oc}}(k) - \frac{k+K}{2}\right) + \frac{|U_{gg}|}{4}\left(1 - 2g_{\text{oc}}(k)\right) + \frac{m_{\text{o}}}{m_{\text{o}} + m_{\text{oc}}} \frac{|U_{gg}|}{8}\left(1 - \bar{g}_{\text{o}}\right) \\
&+ \frac{m_{\text{oc}}}{m_{\text{o}} + m_{\text{oc}}} \frac{|U_{gg}|}{8}\left(1 - g_{\text{oc}}(k)\right) - \frac{m_{\text{oc}}}{m_{\text{o}} + m_{\text{oc}}} \frac{|U_{gg}|}{8} \frac{dg_{\text{oc}}}{dk}(k)
\end{aligned}$$

where dg_{oc}/dk denotes either the left or right derivative of dg_{oc}/dk. The first term is positive because, wheneve the dg_{oc}/dk is not zero, then $g_{\text{oc}}(k) \geq (k+K)/2$. The last term goes to zero uniformly in k. The result then follows by taking limit of the other terms.

B.13 Proof of Lemma 17

Consider a $(0, k)$-type bank that participates to the OTC markets. This bank will enter the market with a possibly non-zero position $\varphi(k)$. More precisely $\varphi(k) = k/2$ if the bank also participates in the centralized market, and $\varphi(k) = 0$ otherwise. The post-trade exposure $g_{\text{o/oc}}(k)$ of such a bank is as follows:

$$(i) \text{ if } \varphi(k) + \overbrace{\frac{m_{\mathrm{o}}}{m_{\mathrm{o}}+m_{\mathrm{oc}}}\mathbb{E}\left[\frac{\max(k,k')}{2}\middle| k' \in X_{\mathrm{o}}\backslash X_{\mathrm{oc}}\right] + \frac{m_{\mathrm{oc}}}{m_{\mathrm{o}}+m_{\mathrm{oc}}}\mathbb{E}\left[\frac{\max(k,k')}{2}\middle| k' \in X_{\mathrm{o}}\right]}^{=\mathbb{E}\left[\frac{\max(k,k')}{2}\middle| k'\in X_{\mathrm{o}}\right]} < \bar{g}_{\mathrm{o}},$$

$$\text{then } g_{\mathrm{o/oc}}(k) = \varphi(k) + \mathbb{E}\left[\frac{\max(k,k')}{2}\middle| k' \in X_{\mathrm{o}}\right];$$

$$(ii) \text{ if } \varphi(k) + \frac{m_{\mathrm{oc}}}{m_{\mathrm{o}}+m_{\mathrm{oc}}}\mathbb{E}\left[\frac{\max(k,k')}{2})\middle| k' \in X_{\mathrm{oc}}\right] \leq \bar{g}_{\mathrm{o}} \leq \varphi(k) + \mathbb{E}\left[\frac{\max(k,k')}{2}\middle| k' \in X_{\mathrm{o}}\right], \text{ then}$$

$$g_{\mathrm{o/oc}}(k) = \bar{g}_{\mathrm{o}};$$

$$(iii) \text{ if } \bar{g}_{\mathrm{o}} < \varphi(k) + \frac{m_{\mathrm{oc}}}{m_{\mathrm{o}}+m_{\mathrm{oc}}}\mathbb{E}\left[\frac{\max(k,k')}{2}\middle| k' \in X_{\mathrm{oc}}\right] < \frac{1}{2}, \text{ then}$$

$$g_{\mathrm{o/oc}}(k) = \varphi(k) + \frac{m_{\mathrm{oc}}}{m_{\mathrm{o}}+m_{\mathrm{oc}}}\mathbb{E}\left[\frac{\max(k,k')}{2}\middle| k' \in X_{\mathrm{oc}}\right];$$

$$(iv) \text{ if } \frac{1}{2} \leq \varphi(k) + \frac{m_{\mathrm{oc}}}{m_{\mathrm{o}}+m_{\mathrm{oc}}}\mathbb{E}\left[\frac{\max(k,k')}{2}\middle| k' \in X_{\mathrm{oc}}\right], \text{ then}$$

$$g_{\mathrm{o/oc}}(k) = \frac{1}{2};$$

First, we can verify that a bank with $k = 0$ and $\omega = 0$ has a post-trade exposure that is at least equal to $\bar{g}_{\mathrm{o}}$ so that case (i) does not apply. Indeed when such a bank participates exclusively in the OTC market, if its post trade exposure was lower than $\bar{g}_{\mathrm{o}}$ it would be equal to $\mathbb{E}\left[k'/2 | \, k' \in X_{\mathrm{o}}\right]$. But we can check that $\mathbb{E}\left[k'/2 | \, k' \in X_{\mathrm{o}}\right] > \bar{g}_{\mathrm{o}}$. Indeed

$$\mathbb{E}\left[\frac{k'}{2}\middle| k' \in X_{\mathrm{o}}\right] = \frac{1}{2}\left(\frac{m_{\mathrm{o}}}{m_{\mathrm{oc}}+m_{\mathrm{o}}}\frac{k^\star}{2} + \frac{m_{\mathrm{oc}}}{m_{\mathrm{oc}}+m_{\mathrm{o}}}\frac{1+k^{\star\star}}{2}\right).$$

Hence, post-trade exposures are given by cases (ii), (iii), and (iv).

B.14 Proof of Lemma 18

The MPV of exclusive entry in the centralized exchange is obvious. In the proof of Lemma 1, we have shown that the full appropriation surplus decomposes as the sum of a quasi-competitive surplus and half the bargaining surplus. The quasi-competitive surplus of the previous bank on the OTC market can be computed as

$$\begin{aligned} Q(0,k,\mathrm{o/oc}) &= U(g_{\mathrm{o/oc}}(k)) - U_g(1/2)\varphi(k) - U_g(g_{\mathrm{o/oc}}(k))(g_{\mathrm{o/oc}}(k) - \varphi(k)) \\ &= \frac{|U_{gg}|}{2}\left[g_{\mathrm{o/oc}}(k)^2 + \varphi(k)(1 - 2g_{\mathrm{o/oc}}(k))\right] \end{aligned}$$

To compute the bargaining surplus of such on the OTC market, we distinguish the different cases (ii), (iii), and (iv), introduced in the proof of Lemma 17. In case (ii)

$$
\begin{aligned}
B(0,k,\mathrm{o/oc}) &= |U_{gg}|\frac{m_{\mathrm{o}}/2}{m_{\mathrm{o}}+m_{\mathrm{oc}}}\,(1-2\bar{g}_{\mathrm{o}})\,\mathbb{E}\left[\left.\frac{\max(k,k')}{2}\right| k'\in X_{\mathrm{o}}\backslash X_{\mathrm{oc}}\right] \\
&+|U_{gg}|\frac{m_{\mathrm{oc}}}{m_{\mathrm{o}}+m_{\mathrm{oc}}}\left(\frac{1}{2}-\bar{g}_{\mathrm{o}}\right)\mathbb{E}\left[\left.\frac{\max(k,k')}{2}\right| k'\in X_{\mathrm{oc}}\right] \\
&= |U_{gg}|\left(\frac{1}{2}-\bar{g}_{\mathrm{o}}\right)\mathbb{E}\left[\left.\frac{\max(k,k')}{2}\right| k'\in X_{\mathrm{o}}\right].
\end{aligned}
$$

In case (iii), we have

$$
\begin{aligned}
B(0,k,\mathrm{o/oc}) &= |U_{gg}|\frac{m_{\mathrm{o}}/2}{m_{\mathrm{o}}+m_{\mathrm{oc}}}\,\left(g_{\mathrm{o/oc}}(k)-\bar{g}_{\mathrm{o}}\right)\mathbb{E}\left[\left.\frac{\max(k,k')}{2}\right| k'\in X_{\mathrm{o}}\backslash X_{\mathrm{oc}}\right] \\
&+|U_{gg}|\frac{m_{\mathrm{o}}/2}{m_{\mathrm{o}}+m_{\mathrm{oc}}}\,\left(1-\bar{g}_{\mathrm{o}}-g_{\mathrm{o/oc}}(k)\right)\mathbb{E}\left[\left.\frac{\max(k,k')}{2}\right| k'\in X_{\mathrm{o}}\backslash X_{\mathrm{oc}}\right] \\
&+|U_{gg}|\frac{m_{\mathrm{oc}}}{m_{\mathrm{o}}+m_{\mathrm{oc}}}\left(\frac{1}{2}-g_{\mathrm{o/oc}}(k)\right)\mathbb{E}\left[\left.\frac{\max(k,k')}{2}\right| k'\in X_{\mathrm{oc}}\right] \\
&= |U_{gg}|\frac{m_{\mathrm{o}}}{m_{\mathrm{o}}+m_{\mathrm{oc}}}\left(\frac{1}{2}-\bar{g}_{\mathrm{o}}\right)\mathbb{E}\left[\left.\frac{\max(k,k')}{2}\right| k'\in X_{\mathrm{o}}\backslash X_{\mathrm{oc}}\right] \\
&+|U_{gg}|\left(\frac{1}{2}-g_{\mathrm{o/oc}}(k)\right)\left(g_{\mathrm{o/oc}}(k)-\varphi(k)\right)
\end{aligned}
$$

In case (iv), we have

$$
B(0,k,\mathrm{o/oc}) = |U_{gg}|\frac{m_{\mathrm{o}}}{m_{\mathrm{o}}+m_{\mathrm{oc}}}\left(\frac{1}{2}-\bar{g}_{\mathrm{o}}\right)\mathbb{E}\left[\left.\frac{\max(k,k')}{2}\right| k'\in X_{\mathrm{o}}\backslash X_{\mathrm{oc}}\right].
$$

Overall, the marginal private value of such a bank is

$$
\mathrm{MPV}(0,k,\mathrm{o/oc}) = Q(0,k,\mathrm{o/oc})+\frac{1}{2}B(0,k,\mathrm{o/oc}).
$$

In case (ii), we have

$$
\begin{aligned}
\mathrm{MPV}(0,k,\mathrm{o/oc}) &= \frac{|U_{gg}|}{2}\left[\bar{g}_{\mathrm{o}}^2+\varphi(k)(1-2\bar{g}_{\mathrm{o}})+\left(\frac{1}{2}-\bar{g}_{\mathrm{o}}\right)\mathbb{E}\left[\left.\frac{\max(k,k')}{2}\right| k'\in X_{\mathrm{o}}\right]\right] \\
&= \frac{|U_{gg}|}{2}\left[\frac{1}{4}-\left(\frac{1}{2}-\bar{g}_{\mathrm{o}}\right)\left(\frac{1}{2}+\bar{g}_{\mathrm{o}}\right)+\varphi(k)(1-2\bar{g}_{\mathrm{o}})+\left(\frac{1}{2}-\bar{g}_{\mathrm{o}}\right)\mathbb{E}\left[\left.\frac{\max(k,k')}{2}\right| k'\in X_{\mathrm{o}}\right]\right] \\
&= \frac{|U_{gg}|}{2}\left[\frac{1}{4}-\left(\frac{1}{2}-\bar{g}_{\mathrm{o}}\right)\left(\frac{1}{2}-\varphi(k)+\overbrace{\bar{g}_{\mathrm{o}}-\varphi(k)-\frac{m_{\mathrm{oc}}}{m_{\mathrm{o}}+m_{\mathrm{oc}}}\mathbb{E}\left[\left.\frac{\max(k,k')}{2}\right| k'\in X_{\mathrm{oc}}\right]}^{>0}\right)\right] \\
&\quad+\frac{|U_{gg}|}{2}\frac{m_{\mathrm{o}}}{m_{\mathrm{o}}+m_{\mathrm{oc}}}\left(\frac{1}{2}-\bar{g}_{\mathrm{o}}\right)\mathbb{E}\left[\left.\frac{\max(k,k')}{2}\right| k'\in X_{\mathrm{o}}\backslash X_{\mathrm{oc}}\right].
\end{aligned}
$$

As in case (ii), we have $g_{\mathrm{o/oc}}(k) = \bar{g}_{\mathrm{o}}$ and $\varphi(k) = 0$ for exclusive participants, or $\varphi(k) = k/2$ for non-exclusive participants, the results hold. In case (iii), we have

$$\begin{aligned}
\mathrm{MPV}(0,k,\mathrm{o/oc}) &= \frac{|U_{gg}|}{2}\left[g_{\mathrm{o/oc}}(k)^2 + \varphi(k)(1-2g_{\mathrm{o/oc}}(k)) + \left(\frac{1}{2} - g_{\mathrm{o/oc}}(k)\right)\left(g_{\mathrm{o/oc}}(k) - \varphi(k)\right)\right] \\
&\quad + \frac{|U_{gg}|}{2}\left[\frac{m_{\mathrm{o}}}{m_{\mathrm{o}}+m_{\mathrm{oc}}}\left(\frac{1}{2} - \bar{g}_{\mathrm{o}}\right)\mathbb{E}\left[\frac{\max(k,k')}{2}\middle| k' \in X_{\mathrm{o}} \backslash X_{\mathrm{oc}}\right]\right] \\
&= \frac{|U_{gg}|}{2}\left[\frac{1}{4} - \left(\frac{1}{2} - g_{\mathrm{o/oc}}(k)\right)\left(\frac{1}{2} - \varphi(k)\right) + \frac{m_{\mathrm{o}}}{m_{\mathrm{o}}+m_{\mathrm{oc}}}\left(\frac{1}{2} - \bar{g}_{\mathrm{o}}\right)\mathbb{E}\left[\frac{\max(k,k')}{2}\middle| k' \in X_{\mathrm{o}} \backslash X_{\mathrm{oc}}\right]\right]
\end{aligned}$$

As in case (ii), we have $g_{\mathrm{o/oc}}(k) = \varphi(k) + \frac{m_{\mathrm{oc}}}{m_{\mathrm{o}}+m_{\mathrm{oc}}}\mathbb{E}\left[\frac{\max(k,k')}{2}\middle| k' \in X_{\mathrm{oc}}\right] \geq \bar{g}_{\mathrm{o}}$ and $\varphi(k) = 0$ for exclusive participants, or $\varphi(k) = k/2$ for non-exclusive participants, the results hold. Finally in case (iv), we have

$$\mathrm{MPV}(0,k,\mathrm{o/oc}) = \frac{|U_{gg}|}{2}\left[\frac{1}{4} + \frac{m_{\mathrm{o}}}{m_{\mathrm{o}}+m_{\mathrm{oc}}}\left(\frac{1}{2} - \bar{g}_{\mathrm{o}}\right)\mathbb{E}\left[\frac{\max(k,k')}{2}\middle| k' \in X_{\mathrm{o}} \backslash X_{\mathrm{oc}}\right]\right],$$

in which case $g_{\mathrm{o/oc}}(k) = 1/2$ and the result obviously holds.

C Flat and increasing spots and the atom property

C.1 Some elementary results about flat and increasing spots

In many examples, we work with post-trade exposures that are weakly increasing and continuous functions:

$$\begin{aligned}
g\,:&\, X \to Y \\
&\, x \mapsto g(x)
\end{aligned}$$

where X and $Y \equiv g(X)$ are compact intervals. The interval X indexes the heterogeneity between banks, either endowment or capacity depending on the example.

Our objective in this section is to derive some elementary mathematical properties of the flat spots of g. Formally, we define the set value function:

$$\begin{aligned}
F\,:&\, Y \to \mathcal{P}(X) \\
&\, y \mapsto g^{-1}(y),
\end{aligned}$$

which maps any value in the range of g into its inverse image. We first show that the sets $F(y)$ are all compact intervals.

Lemma 23. *For all $y \in Y$, $F(y) = [\inf F(y), \sup F(y)]$.*

Proof. Because g is continuous, it follows that $F(y)$ is closed, hence $\{\inf F(y), \sup F(y)\} \subseteq F(y)$ or, equivalently:

$$g(\inf F(y)) = g(\sup F(y)) = y.$$

Since g is increasing it follows that all $g(x) = y$ for all $x \in [\inf F(y), \sup F(y)]$. $\square$

Lemma 24. *The set of flat spot of g, i.e. the set of y such that $\inf F(y) < \sup F(y)$ is countable.*

Proof. Indeed, one can associate each flat spot with a unique rational number in the interval $(\inf F(y), \sup F(y))$. $\square$

A corollary of this result is:

Lemma 25. *Take any $y < y'$ in Y. Then there exists $y'' \in (y, y')$ such that $F(y'')$ is a singleton.*

Proof. Indeed, if $y < y'$, the set (y, y') is not countable and so must contain some y'' with the desired property. $\square$

Another corollary works as follows. Consider any point x at the upper-end of a flat spot, i.e. $x = \sup F(g(x))$. Then for all $x' > x$, $g(x') > g(x)$. If we let $y = g(x)$ and $y' = g(x')$, the previous Lemma shows that there is a $x'' \in (x, x')$ such that $F(g(x''))$ is a singleton. The sequential version of this result is:

Lemma 26. *Let $x \in [\inf X, \sup X)$ such that $x = \sup F(g(x))$ and $g(x) < \sup Y$. Then there is a strictly decreasing sequence $x_n \to x$ such that $F(g(x_n))$ is a singleton.*

C.2 The atom property with exclusive participation

In this section, we explore the generality of the atom property with exclusive participation. We consider a symmetric economy with two endowment types, $\omega \in \{0, 1\}$ and heterogeneous capacities, $k \in [\underline{k}, \overline{k}]$. We only consider the banks who participate in the OTC market.

C.2.1 Some results about atoms

We let $g(k)$ denote the post-trade exposure of a bank with capacity k and endowment $\omega = 0$. We know from Lemma 11 that $g(k)$ is weakly increasing and continuous, with $g(\overline{k}) \leq 1/2$.

Consider any point k such that $g(k)$ is strictly increasing. Then, to the right of k, $g(k) < g(k') \leq 1/2$. Hence, a bank with capacity k buys up to capacity from all banks with $k' > k$ and $\omega = 0$, and from all banks with $\omega = 1$. Conversely, it will sell up to capacity to all banks with $k' < k$ and $\omega = 0$.

Hence, the sales to $k' < k$ and $\omega = 0$ cancel out with the purchase from $k' < k$ and $\omega = 1$, implying that the post-trade exposure of the bank can be written:

$$g(k) = \int_k^{\bar{k}} \Gamma(k, k')\, dN(k' \,|\, \mathrm{o}). \tag{44}$$

We can extend this formula to flat spots: as in Section C.1, we use the set function F to describe the flat spots of the function g: namely, for any y in the range of g, we let $F(y) \equiv g^{-1}(y)$. Take any k such that $\sup F(g(k)) < \overline{k}$. Then since g increases strictly to the right of $\sup F(g(k))$, it must be that $g(k) < 1/2$. Now from Lemma 26 it follows that there is a strictly decreasing sequence $k_n \to k$ such that $F(g(k_n)) = \{k_n\}$ and hence g is strictly increasing at k_n. Therefore, formula (44) holds at each k_n and, by continuity, it holds at $\sup F(g(k))$. This leads to:

Lemma 27. *If* $\sup F(g(k)) < \overline{k}$, *then*

$$g(k) = \int_{\sup F(g(k))}^{\bar{k}} \Gamma(\sup F(g(k)), k')\, dN(k' \,|\, \mathrm{o}).$$

Likewise, if $\inf F(g(k)) > \underline{k}$,

$$g(k) = \int_{\inf F(g(k))}^{\bar{k}} \Gamma(\inf F(g(k)), k')\, dN(k' \,|\, \mathrm{o}). \tag{45}$$

Note that a corollary of these formula is

Lemma 28. *There must be a flat spot at the top, i.e.* $F(g(\bar{k})) < \bar{k}$.

Indeed, if it were not the case, then (45) would imply that $g(\bar{k}) = 0$, which is impossible.

Let $\hat{k} = \inf F(g(\overline{k}))$ denote the lower bound of the flat spot $F(g(\overline{k}))$. The aggregate trades of $k > \hat{k}$, $\omega = 0$ banks with $k < \hat{k}$ banks net out to zero: indeed, they sell at full capacity, to all banks with $k < \hat{k}$ and $\omega = 0$, and buy, at full capacity again, from all banks with $k < \hat{k}$ and $\omega = 1$. The aggregate trades of $k > \hat{k}$, $\omega = 0$ banks with $k > \hat{k}$, $\omega = 1$ banks are less than

$$\frac{1}{2} \int_{\hat{k}}^{\overline{k}} \int_{\hat{k}}^{\overline{k}} \Gamma(k', k'')\, dN(k' \,|\, o)\, dN(k'' \,|\, o),$$

with an equality if $g(\overline{k}) < 1/2$. Taken together, we obtain:

$$g(\overline{k}) = \min \left\{ \frac{1}{2}, \frac{1}{2} \int_{\hat{k}}^{\overline{k}} \int_{\hat{k}}^{\overline{k}} \Gamma(k', k'')\, dN(k' \,|\, k \geq \hat{k}, \mathrm{o})\, dN(k'' \,|\, \mathrm{o}) \right\}.$$

Using (45) we obtain that, if $\hat{k} > \underline{k}$, then

$$\int_{\hat{k}}^{\overline{k}} \Gamma(\hat{k}, k')\, dN(k' \,|\, k \geq \hat{k}, \mathrm{o}) = \min\left\{\frac{1}{2}, \frac{1}{2}\int_{\hat{k}}^{\overline{k}}\int_{\hat{k}}^{\overline{k}} \Gamma(k', k'')\, dN(k' \,|\, k \geq \hat{k}, \mathrm{o})\, dN(k'' \,|\, k \geq \hat{k}, \mathrm{o})\right\}.$$

This delivers a sufficient condition for $\hat{k} = \underline{k}$, that is, for the atom property to hold:

Lemma 29. *A sufficient condition for the atom property to hold is that:*

$$\mathbb{E}\left[\Gamma(\hat{k}, k) \,\middle|\, k \geq \hat{k}, \mathrm{o}\right] > \frac{1}{2}\mathbb{E}\left[\Gamma(k, k') \,\middle|\, (k, k') \geq \hat{k}, \mathrm{o}\right]. \tag{46}$$

for all $\hat{k} \in [\underline{k}, \overline{k}]$.

C.2.2 Examples

Max capacity constraint: $\Gamma(k, k') = \max\{k, k'\}$**.** We can verify that the max specification satisfies the sufficient condition. Indeed, consider any distribution N on the support $[\hat{k}, \overline{k}]$. The left-hand side of (46) is equal to

$$\int_{\hat{k}}^{\overline{k}} k\, dN(k).$$

To calculate the right-hand side, notice that the CDF of the maximum is $N(k)^2$. Hence, the right hand side writes:

$$\int_{\hat{k}}^{\overline{k}} k\, N(k) dN(k).$$

Since $N(k) < 1$ on $[\hat{k}, \overline{k})$, the result follows.

Separable capacity constraint: $\Gamma(k, k') = f(k) + f(k')$**.** Then the left-hand side of (46) is

$$f(\hat{k}) + \mathbb{E}\left[f(k)\right].$$

The right-hand side is

$$\mathbb{E}\left[f(k)\right],$$

which is greater than the left-hand side.

Submodular capacity constraint. This seems to encompass both the max and the separable cases. The definition of submodularity, according to Bach (2019) is that, for all (k_1, k_1') and (k_2, k_2'):

$$\Gamma(k_1, k_1') + \Gamma(k_2, k_2') \geq \Gamma(\max\{k_1, k_2\}, \max\{k_1', k_2'\}) + \Gamma(\min\{k_1, k_2\}, \min\{k_1', k_2'\}).$$

If Γ is twice differentiable, it is equivalent to $\partial^2\Gamma/\partial k \partial k' \leq 0$.

To see why submodularity implies the atom property, let $(k_1, k_1') = (k', \hat{k})$ and $(k_2, k_2') = (\hat{k}, k'')$, for $(k', k'') \in [\hat{k}, \overline{k}]^2$. The submodularity condition implies that:

$$\Gamma(k', \hat{k}) + \Gamma(\hat{k}, k'') \geq \Gamma(k', k'') + \Gamma(\hat{k}, \hat{k}) > \Gamma(k', k''),$$

if $\Gamma(\hat{k}, \hat{k}) > 0$. Using symmetry we have $\Gamma(k', \hat{k}) = \Gamma(\hat{k}, k')$. Taking expectations on both sides with respect to $N(k \,|\, \mathrm{o}, k \geq \hat{k})$ leads to the desired inequality.

D Exclusive participation under the "max" specification

In this section, we consider a setup very similar to the one introduced in Section 4.1. We assume that capacities are heterogeneous across banks: they are distributed according to a generic continuous density $f(k)$ over the compact interval $[0, \infty)$. We also assume that risk-sharing needs are the same for all banks: namely, the endowment distribution has just two points, $\omega = 0$, or $\omega = 1$, with equal probability, and is independent from capacities. However in this section we work under the "max" specification, that is

$$\Gamma(k, k') = \max\{k, k'\}.$$

According to Section C results, we know that the atom property holds. That is, banks with endowments $\omega = 0$, or $\omega = 1$, that participate in the OTC market end up with a post trade exposure respectively equal to g or $1 - g$, where

$$g = \frac{1}{2}\mathbb{E}\left[\max\left\{k', k''\right\} \mid (k', k'') \in X_{\mathrm{o}}^2\right],$$

where X_{o} is given subset of (symmetric) banks, indexed by their trading capacities, that participate in the OTC market.

Consider a bank with endowment $\omega = 0$, and capacity k. Its quasi-competitive surplus is equal to

$$Q(0, k, \mathrm{o}) = U(g) - U(0) - U_g[g]g = \frac{|U_{gg}|}{2}g^2 \equiv Q(g).$$

Such a bank makes zero bargaining surplus with other banks with similar endowments and makes a $|U_{gg}|(1-2g)$ per unit surplus with $\omega = 1$ banks. Its bargaining surplus is therefore

$$B(0,k,\mathrm{o}) = \frac{1}{2}|U_{gg}|(1-2g)\mathbb{E}\left[\max\{k,k'\} \mid k' \in X_{\mathrm{o}}\right].$$

By symmetry we have $Q(1,k,\mathrm{o}) = Q(0,k,\mathrm{o}) = Q(g)$ and $B(1,k,\mathrm{o}) = B(0,k,\mathrm{o})$, and we can compute the marginal private value of entry in the OTC market of a bank with capacity k as

$$\mathrm{MPV}(k,\mathrm{o}) = \frac{|U_{gg}|}{2}\left(g^2 + \left(\frac{1}{2} - g\right)\mathbb{E}\left[\max\{k,k'\} \mid k' \in X_{\mathrm{o}}\right]\right).$$

One can also compute the marginal private value of entry in the centralized market of bank with capacity k as

$$\mathrm{MPV}(k,\mathrm{c}) = \frac{|U_{gg}|}{2}\min\left\{\max\{k,K\},\frac{1}{2}\right\}\left(1 - \min\left\{\max\{k,K\},\frac{1}{2}\right\}\right).$$

Lemma 30. *For any participation pattern,* $\mathrm{MPV}(k,\mathrm{o})$ *is convex and increasing on* $[0,\infty)$*, and* $\mathrm{MPV}(k,\mathrm{c})$ *is flat on* $[0,K]$*, and then increasing and concave on* $[K,\infty)$*. Therefore* $\mathrm{MPV}(k,\mathrm{o})$ *and* $\mathrm{MPV}(k,\mathrm{c})$ *cross at most twice on* $[K,\infty)$*.*

Proof. Denote $F(k|\mathrm{o})$ the conditional distribution of capacities in the OTC market. Then,

$$\mathbb{E}\left[\max\{k,k'\} \mid k' \in X_{\mathrm{o}}\right] = kF(k|\mathrm{o}) + \int_k^{\infty} k' dF(k'|\mathrm{o}).$$

Taking the derivative of the former leads to

$$F(k|\mathrm{o}) + kF'(k|\mathrm{o}) - kF'(k|\mathrm{o}) = F(k|\mathrm{o}),$$

which is increasing in k. $\mathrm{MPV}(k,\mathrm{o})$ is therefore convex and obviously increasing.

$\mathrm{MPV}(k,\mathrm{c})$ is S-shaped and increasing because it is flat on $[0,K]$, increasing and concave on $[K,1/2]$, and flat again on $[1/2,\infty)$. □

Lemma 31. *Consider a general distribution of capacities and a general exclusive participation pattern. Then,* $\mathrm{MPV}(k,\mathrm{o}) - \mathrm{MPV}(k,\mathrm{c})$ *is negative for some* k*, in particular for* k_g *such that* $\mathrm{MPV}(k_g,\mathrm{o}) = \frac{|U_{gg}|}{2}g(1-g)$*. Moreover* $B(k_g,\mathrm{o}) = \bar{B}$*.*

Proof. Consider k_g such that $\mathbb{E}\left[\max\{k_g,k'\} \mid k' \in X_{\mathrm{o}}\right] = 2g$. Then, we have

$$\mathrm{MPV}(k_g,\mathrm{o}) = \frac{|U_{gg}|}{2}g(1-g).$$

If $k_g > g$, then we necessarily have $\mathrm{MPV}(k_g,\mathrm{o}) \leq \mathrm{MPV}(k_g,\mathrm{c})$.

Proving that $k_g \geq g$ is then equivalent to show that

$$\begin{aligned}
&\mathbb{E}\left[\max\left\{g, k'''\right\} \mid k''' \in X_{\mathrm{o}}\right] \leq 2g \\
\Leftrightarrow\ &\mathbb{E}\left[\max\left\{\frac{1}{2}\mathbb{E}\left[\max\left\{k', k''\right\} \mid (k', k'') \in X_{\mathrm{o}}^2\right], k'''\right\} \mid k''' \in X_{\mathrm{o}}\right] \leq \mathbb{E}\left[\max\left\{k', k'''\right\} \mid (k', k''') \in X_{\mathrm{o}}^2\right]
\end{aligned}$$

We know that

$$\mathbb{E}\left[\max\left\{k', k''\right\} \mid (k', k'') \in X_{\mathrm{o}}^2\right] \leq 2\mathbb{E}\left[k' \mid k' \in X_{\mathrm{o}}\right],$$

thus we have

$$\mathbb{E}\left[\max\left\{\frac{1}{2}\mathbb{E}\left[\max\left\{k', k''\right\} \mid (k', k'') \in X_{\mathrm{o}}^2\right], k'''\right\} \mid k''' \in X_{\mathrm{o}}\right] \leq \mathbb{E}\left[\max\left\{\mathbb{E}\left[k' \mid k' \in X_{\mathrm{o}}\right], k'''\right\} \mid k''' \in X_{\mathrm{o}}\right]$$

For a given k''', as $\max\{k', k'''\}$ is a convex function of k', thanks to Jensen's Inequality we have

$$\mathbb{E}\left[\max\left\{k', k'''\right\} \mid k' \in X_{\mathrm{o}}\right] \geq \max\left\{\mathbb{E}\left[k' \mid k' \in X_{\mathrm{o}}\right], k'''\right\},$$

and then by taking the expectation w.r.t k'''

$$\mathbb{E}\left[\max\left\{k', k'''\right\} \mid (k', k''') \in X_{\mathrm{o}}^2\right] \geq \mathbb{E}\left[\max\left\{\mathbb{E}\left[k' \mid k' \in X_{\mathrm{o}}\right], k'''\right\} \mid k''' \in X_{\mathrm{o}}\right].$$

Hence the result.

In addition one can see that taking the average MPV gives on the one hand $Q(g) + \bar{B}/2$, and on the other hand

$$\frac{|U_{gg}|}{2}\left(g^2 + \left(\frac{1}{2} - g\right) 2g\right) = \frac{|U_{gg}|}{2} g(1-g) = \mathrm{MPV}(k_g, \mathrm{o}) = Q(g) + \frac{B(k_g, \mathrm{o})}{2}.$$

□

Proposition 7. *When $C(\mathrm{o}) = C(\mathrm{c}) = 0$, if an equilibrium exists then the set of banks that participate in the OTC market, X_{o}, and the set of banks that participate in the centralized market, X_{c}, are defined as*

$$X_{\mathrm{o}} = [k^{\star\star\star}, k^{\star\star}] \cup [k^{\star}, \infty), \text{ and } X_{\mathrm{c}} = [0, k^{\star\star\star}) \cup (k^{\star\star}, k^{\star}),$$

where thresholds, $k^{\star\star\star}$, $k^{\star\star}$ and $k^{\star}$, are pined down by equilibrium conditions,

(a) $k^{\star\star\star} = \sup\{k \in [0,K], \mathrm{MPV}(k,\mathrm{o}) \leq \mathrm{MPV}(K,\mathrm{c})\}$, *or* $k^{\star\star\star} = 0$ *if* $\mathrm{MPV}(0,\mathrm{o}) > \mathrm{MPV}(K,\mathrm{c})$

(b) $\mathrm{MPV}(k^{\star\star},\mathrm{o}) = \mathrm{MPV}(k^{\star\star},\mathrm{c})$ *and* $\frac{1}{2} \geq k^{\star\star} \geq K$,

(c) $\mathrm{MPV}(k^{\star},\mathrm{o}) = \mathrm{MPV}(k^{\star},\mathrm{c})$ *and* $k^{\star} \geq k^{\star\star}$.

Moreover $B(k^{\star\star},\mathrm{o}) \leq \bar{B} \leq B(k^{\star},\mathrm{o})$.[8]

Proof. According to Lemma 31, the set of OTC banks cannot be convex, because the bank k_g does not participate in the OTC market but still above and below capacities of banks participating in the OTC market. So according to Lemma 30, there must be at least two crossing points $k^{\star\star}$ and $k^{\star} > k^{\star\star}$ between $\mathrm{MPV}(k,\mathrm{o})$ and $\mathrm{MPV}(k,\mathrm{c})$, so that $\mathrm{MPV}(k,\mathrm{c}) > \mathrm{MPV}(k,\mathrm{o})$ on $(k^{\star\star},k^{\star})$, and so that $k_g \in (k^{\star\star},k^{\star})$. These two crossing points must necessarily be on the concave branch of $\mathrm{MPV}(k,\mathrm{c})$, that is $[K,\infty)$, and $k^{\star\star}$ must be on its strictly increasing part, that is $[K,1/2]$.

On top of those necessary conditions, it is possible that $\mathrm{MPV}(k,\mathrm{o})$ and $\mathrm{MPV}(k,\mathrm{c})$ coincide again on the segment $[0,K]$. Define $k^{\star\star\star}$ such that $\mathrm{MPV}(k,\mathrm{o}) > \mathrm{MPV}(k,\mathrm{c})$ on $(k^{\star\star\star},K]$, and $\mathrm{MPV}(k,\mathrm{o}) \leq \mathrm{MPV}(k,\mathrm{c})$ otherwise. Notice that on $[0,k^{\star\star\star})$, $\mathrm{MPV}(k,\mathrm{o})$ must be flat as the lowest capacity in the OTC market would be $k^{\star\star\star}$. Therefore we would have $\mathrm{MPV}(k,\mathrm{o}) = \mathrm{MPV}(k,\mathrm{c}) = \mathrm{MPV}(K,\mathrm{c})$ on $[0,k^{\star\star\star})$. Banks on this segment would therefore be indifferent between participating in the OTC or the centralized. However in equilibrium they would all participate in the centralized market.

In order to restore strict preference in participation, one could introduce an arbitrarily small mass of $k=0$-banks to participate in the OTC market. In such case banks in $[0,k^{\star\star\star})$ would strictly prefer to participate in the centralized market. We use this variation when we solve for the equilibrium existence, so that $k^{\star\star\star}$ can be pinned down as a locally unique indifference threshold.

$B(k^{\star\star},\mathrm{o}) \leq \bar{B} \leq B(k^{\star},\mathrm{o})$ is obtained according to Lemma 31. $\square$

Lemma 32. *In equilibrium, $k^{\star} \geq 1/2$.*

Proof. Denote with $|U_{gg}|a/2$ the slope of the affine portion of $\mathrm{MPV}(.,\mathrm{o})$ between on $(k^{\star\star},k^{\star})$, that is

$$a = \left(\frac{1}{2} - g\right) \frac{F(k^{\star\star}) - F(k^{\star\star\star})}{1 - F(k^{\star}) + F(k^{\star\star}) - F(k^{\star\star\star})} \leq \frac{1}{2} - g$$

To prove this result, it is sufficient to show that

$$h(k^{\star\star}) = k^{\star\star}(1-k^{\star\star}) + \left(\frac{1}{2} - g\right)\left(\frac{1}{2} - k^{\star\star}\right) < \frac{1}{4}.$$

[8] In this section we "inversely" rank thresholds as $k^{\star} > k^{\star\star} > k^{\star\star\star}$ because the third one may not exist, i.e $k^{\star\star\star} = 0$.

which ensures that $\mathrm{MPV}(1/2, \mathrm{o}) = \mathrm{MPV}(k^{\star\star}, \mathrm{o/c}) + |U_{gg}|a(1/2 - k^{\star\star})/2 < |U_{gg}|/8$. One can compute the derivative of h as

$$h'(k) = 1 - 2k - \frac{1}{2} + g = \frac{1}{2} + g - 2k > 0, \ \forall k \in [0, g].$$

Remember that Lemma 31 shows that there is a $k_g \geq g$ such that $\mathrm{MPV}(k_g, \mathrm{o}) = \frac{|U_{gg}|}{2} g(1-g) = \mathrm{MPV}(g, \mathrm{c}) < \mathrm{MPV}(k_g, \mathrm{c})$. Then we necessarily have that $k_g \in (k^{\star\star}, k^{\star})$. Since $\mathrm{MPV}(k_g, \mathrm{o}) \geq \mathrm{MPV}(k^{\star\star}, \mathrm{o}) = \mathrm{MPV}(k^{\star\star}, \mathrm{c})$, we have $k^{\star\star} \leq g$. Then we obtain that

$$h(k^{\star\star}) \leq h(g) = g(1-g) + \left(\frac{1}{2} - g\right)^2 = \frac{1}{4}.$$

□

Example. We consider a case in which capacities are uniformly distributed over the segment $[0,1]$, that is $f(k) = \mathbb{I}_{\{k \in [0,1]\}}$. In addition we assume that $C(\mathrm{o}) = C(\mathrm{c}) = 0$ and $C(\mathrm{oc}) = \infty$ in order to induce exclusive participation. Furthermore, we consider the case in which K is low enough so that $k^{\star\star\star} = 0$.

Consider the three variables, $\{g, k^{\star}, k^{\star\star}\}$, equations system:

$$\begin{aligned}
&g = \frac{2 + 2\left(k^{\star\star}\right)^3 - 3k^{\star} + \left(k^{\star}\right)^3 + 3k^{\star\star}\left(1 - (k^{\star})^2\right)}{6\left(1 - k^{\star} + k^{\star\star}\right)^2} \\
&\frac{1-2g}{2} \frac{k^{\star\star}\left(k^{\star} - k^{\star\star}\right)}{1 - k^{\star} + k^{\star\star}} = \frac{1}{4} - k^{\star\star}\left(1 - k^{\star\star}\right) \\
&g^2 + \frac{1-2g}{4} \frac{2\left(k^{\star\star}\right)^2 + 1 - \left(k^{\star}\right)^2}{1 - k^{\star} + k^{\star\star}} = k^{\star\star}\left(1 - k^{\star\star}\right)
\end{aligned}$$

such that $k^{\star} > k^{\star\star}$, $k^{\star\star} > 0$, and $k^{\star\star} < 1/2$. Our aim is to show numerically (with *Mathematica*) that the former solution exists and is unique, and that if

$$0 \leq K \leq \frac{1 - \sqrt{1 - 4\left(g^2 + \frac{1-2g}{4} \frac{(k^{\star\star})^2 + 1 - (k^{\star})^2}{1 - k^{\star} + k^{\star\star}}\right)}}{2},$$

then there exists a unique equilibrium in which banks with $k \in [0, k^{\star\star}] \cup [k^{\star}, 1]$ enter exclusively the OTC market, and vice-versa, banks with $k \in (k^{\star\star}, k^{\star})$ enter exclusively the centralized market. Conditional on $\omega = 0$ and $\pi = \mathrm{o}$, all banks obtain the same post-trade exposure g. Symmetrically, all banks obtain the same post-trade exposure $1 - g$ conditional on $\omega = 1$ and $\pi = \mathrm{o}$. Bilateral trades depend on capacity k. Small-k banks trade like customers: they tend to buy from all banks. Large-k banks trade like dealers: they tend to buy from $\omega = 1$ banks and sell to $\omega = 0$ banks.

Conditional on $\pi = \text{c}$, post-trade exposures may or may not depend on k: $g(0,k,\text{c}) = \min\left\{k, \frac{1}{2}\right\}$ and $g(1,k,\text{c}) = 1 - \min\left\{k, \frac{1}{2}\right\}$.

Proof. Using the fact that k is distributed uniformly on $[0,1]$,

$$\mathbb{E}\left[\max\{k,k'\} \,|\, k' \in X_{\text{o}}\right] = \begin{cases} \frac{k^2+(k^{\star\star})^2+1-(k^\star)^2}{2(1-k^\star+k^{\star\star})} & \text{if } k \le k^{\star\star}, \\ \frac{2kk^{\star\star}+1-(k^\star)^2}{2(1-k^\star+k^{\star\star})} & \text{if } k^{\star\star} < k < k^\star, \\ \frac{1+k^2-2k(k^\star-k^{\star\star})}{2(1-k^\star+k^{\star\star})} & \text{if } k^\star \le k. \end{cases}$$

Using this, and that $g = \mathbb{E}\left[\max\{k',k''\} \,|\, (k',k'') \in X_{\text{o}}^2\right]/2$, one obtains the first equation in the system, where $g \equiv g(0,k,\text{o})$. This is one of the equations that connect together g, $k^\star$, and $k^{\star\star}$ that are all determined in equilibrium. The remaining two equations will come from the indifference conditions of the two marginal banks with the capacities $k^\star$ and $k^{\star\star}$. Recall that an OTC bank's MPV is equal to its full appropriation surplus minus half of its bargaining surplus. With atom property, it is equal to

$$\text{MPV}(k,\text{o}) = \frac{|U_{gg}|}{2}g^2 + \frac{|U_{gg}|}{4}(1-2g)\,\mathbb{E}\left[\max\{k,k'\} \,|\, k' \in X_{\text{o}}\right].$$

In the centralized market, the MPV is equal to

$$\text{MPV}(k,\text{c}) = \frac{|U_{gg}|}{2}\min\left\{\max\{k,K\}, \frac{1}{2}\right\}\left(1 - \min\left\{\max\{k,K\}, \frac{1}{2}\right\}\right).$$

We focus on the case of sufficiently small K such that no bank benefits from the centralized market capacity. Thus, $\max\{k^\star,K\} = k^\star$ and $\max\{k^{\star\star},K\} = k^{\star\star}$.

Using the uniformity of k, the indifference conditions are

$$\begin{aligned} \frac{|U_{gg}|}{2}g^2 + \frac{|U_{gg}|}{4}(1-2g)\frac{2k^\star k^{\star\star}+1-(k^\star)^2}{2(1-k^\star+k^{\star\star})} &= \frac{|U_{gg}|}{2}\min\left\{k^\star, \frac{1}{2}\right\}\left(1-\min\left\{k^\star,\frac{1}{2}\right\}\right) \\ \frac{|U_{gg}|}{2}g^2 + \frac{|U_{gg}|}{4}(1-2g)\frac{2(k^{\star\star})^2+1-(k^\star)^2}{2(1-k^\star+k^{\star\star})} &= \frac{|U_{gg}|}{2}\min\left\{k^{\star\star}, \frac{1}{2}\right\}\left(1-\min\left\{k^{\star\star},\frac{1}{2}\right\}\right). \end{aligned}$$

Because $k^{\star\star} < k^\star$, the LHS of the first equation is larger than the LHS of the second equation. Thus, these two equations can be satisfied simultaneously only if $k^{\star\star} < 1/2$. Using this fact, subtracting the second equation from the first one, and after cancellations,

$$\begin{aligned} \frac{1-2g}{2}\frac{k^{\star\star}(k^\star-k^{\star\star})}{2(1-k^\star+k^{\star\star})} &= \min\left\{k^\star,\frac{1}{2}\right\}\left(1-\min\left\{k^\star,\frac{1}{2}\right\}\right) - k^{\star\star}(1-k^{\star\star}) \\ g^2 + \frac{1-2g}{2}\frac{2(k^{\star\star})^2+1-(k^\star)^2}{2(1-k^\star+k^{\star\star})} &= k^{\star\star}(1-k^{\star\star}). \end{aligned}$$

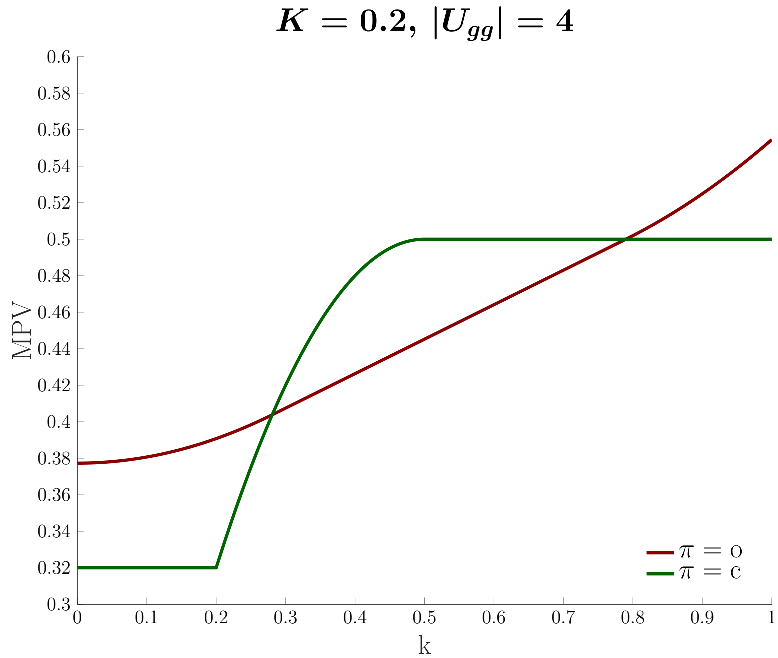

Figure 9: The MPV of centralized (green curve) and OTC market participation (red curve, as functions of capacity, k.

These two equations combined with the first equation stated in the Proposition pin down g, $k^\star$, and $k^{\star\star}$. *Mathematica* finds that under the restrictions $k^\star > k^{\star\star}$, $k^{\star\star} > 0$, and $k^{\star\star} < 1/2$, the solution is unique: $g = 0.334903$, $k^\star = 0.789951$, and $k^{\star\star} = 0.280706$. Because $k^\star > 1/2$ in the unique solution, we can get rid of the min operator in the equation system for simplicity.

What remains is to verify that the MPVs implied by g, $k^\star$, and $k^{\star\star}$ are consistent with the conjectured participation patterns, as illustrated in Figure 9:

$$MPV(k, \mathrm{o}) - MPV(k, \mathrm{c}) \geq 0 \text{ for } k \in [0, k^{\star\star}] \cup [k^\star, 1]$$
$$MPV(k, \mathrm{c}) - MPV(k, \mathrm{o}) \geq 0 \text{ for } k \in (k^{\star\star}, k^\star)$$

For $k \in [0, k^{\star\star}]$,

$$\begin{aligned} & MPV(k, \mathrm{o}) - MPV(k, \mathrm{c}) \\ & \quad = \frac{|U_{gg}|}{2} g^2 + \frac{|U_{gg}|}{4} (1 - 2g) \frac{k^2 + (k^{\star\star})^2 + 1 - (k^\star)^2}{2(1 - k^\star + k^{\star\star})} - \frac{|U_{gg}|}{2} \max\{k, K\} (1 - \max\{k, K\}) . \end{aligned}$$

We should guarantee that this difference is weakly positive for all $k \in [0, k^{\star\star}]$. It is easy to notice that it is increasing in $k < K$ and decreasing in $k \geq K$. It is weakly positive at $k = k^{\star\star}$ by the indifference condition. Thus, it suffices to impose that it is weakly positive at $k = 0$:

$$g^2 + \frac{1-2g}{2} \frac{(k^{\star\star})^2 + 1 - (k^\star)^2}{2(1-k^\star+k^{\star\star})} - K(1-K) \geq 0,$$

which holds when

$$K \leq \frac{1 - \sqrt{1 - 4\left(g^2 + \frac{1-2g}{4} \frac{(k^{\star\star})^2+1-(k^\star)^2}{1-k^\star+k^{\star\star}}\right)}}{2} = 0.2524.$$

This is the first necessary condition stated in the Proposition. For $k \in [k^\star, 1]$,

$$\text{MPV}(k, \text{o}) - \text{MPV}(k, \text{c}) = \frac{|U_{gg}|}{2} g^2 + \frac{|U_{gg}|}{4}(1-2g)\frac{1+k^2-2k(k^\star-k^{\star\star})}{2(1-k^\star+k^{\star\star})} - \frac{|U_{gg}|}{8}.$$

This MPV difference is strictly increasing in $k \in [k^\star, 1]$, and so, the indifference condition guarantees that it is weakly positive for all $k \in [k^\star, 1]$. For $k \in (k^{\star\star}, k^\star)$,

$$\begin{aligned}
&\text{MPV}(k, \text{c}) - MPV(k, \text{o}) \\
&\quad = \frac{|U_{gg}|}{2} \min\left\{k, \frac{1}{2}\right\}\left(1 - \min\left\{k, \frac{1}{2}\right\}\right) - \frac{|U_{gg}|}{2} g^2 - \frac{|U_{gg}|}{4}(1-2g)\frac{2kk^{\star\star}+1-(k^\star)^2}{2(1-k^\star+k^{\star\star})}.
\end{aligned}$$

This MPV difference is hump-shaped in $k \in (k^{\star\star}, k^\star)$, and so, the two indifference conditions guarantee that it is weakly positive for all $k \in (k^{\star\star}, k^\star)$. $\square$