Online Appendix

	3 months	6 months	9 months	12 months	18 months	36 months	60 months	120 months	180 months	360 months	600 months
						ι/y (%)					
mean	0.84	0.92	1.93	2.11	1.05	1.38	1.37	1.60	1.13	0.92	1.14
sd	0.76	0.93	1.23	1.85	0.85	1.16	1.15	1.85	1.34	1.15	1.02
min	0.00	0.00	0.02	0.00	0.01	0.00	0.01	0.00	0.01	0.00	0.05
p50	0.68	0.70	2.30	1.23	1.09	1.19	1.13	1.22	0.73	0.74	1.18
max	4.26	5.34	4.15	5.89	3.38	5.04	5.76	11.70	8.11	8.00	3.23
N	189	222	86	301	130	254	295	332	123	139	8
						Markup on Marginal Price (Basis Points)					
mean	1.42	3.75	1.24	5.72	5.39	7.64	11.72	6.31	6.96	30.73	.
sd	3.19	8.93	3.22	14.89	42.92	66.32	84.42	101.25	100.58	158.69	.
min	-3.91	-4.85	-6.39	-16.89	-254.30	-300.36	-409.73	-375.07	-301.40	-493.36	.
p50	.77	1.47	.13	2.32	3.81	4.52	15.28	18.26	13.70	28.34	.
max	23.08	71.64	11.50	119.09	162.67	367.80	433.98	341.91	332.16	716.15	.
N	112	125	44	170	87	151	161	167	62	69	0
						Markup on Weighted Average Above Marginal (WAAM) Price (Basis Points)					
mean	1.05	2.86	0.09	4.51	2.04	-18.84	-22.36	-32.82	-42.02	-36.46	73.04
sd	3.26	7.73	3.26	15.38	51.67	84.507	106.35	117.63	131.83	174.83	219.75
min	-7.55	-12.03	-17.87	-25.79	-306.65	-414.68	-596.55	-467.97	-437.35	-513.13	-207.78
p50	0.35	0.95	-0.3	1.53	3.30	-2.87	-6.67	-19.21	-22.16	-27.54	62.25
max	23.09	67.93	9.99	119.09	162.67	347.73	427.35	328.52	302.31	676.14	411.08
N	189	220	86	301	130	254	295	332	123	139	8
						Markup on Average Price (Basis Points)					
mean	0.86	2.75	0.42	3.43	0.73	-0.465	0.43	-10.91	-10.34	14.64	.
sd	2.91	8.38	2.96	14.55	43.31	67.30	88.30	107.91	102.10	164.02	.
min	-4.76	-11.42	-7.85	-25.27	-265.13	-315.53	-445.58	-430.75	-327.61	-512.18	.
p50	-0.32	0.86	-0.26	1.26	1.35	1.38	1.48	0.40	4.91	13.82	.
max	19.70	69.17	10.10	107.47	144.49	348.27	430.44	333.24	306.29	678.23	.
N	110	122	44	165	83	145	154	153	58	64	0

Table 2: Descriptive Statistics

The descriptive statistics correspond to the full sample. $(\iota/y)_{\tau,t}$ is computed as the total issuance at the auction of date $t \in \mathcal{M}$ divided by the monthly GDP at the auction month. The average price is constructed as the weighted average between the marginal and the WAAM, where the weights are given by the fraction assigned to bids above and below the marginal. The total number N corresponds to the total number of auctions for each of the maturity categories.

Variables	(1) Whole sample	(2) Until 2010	(3) From 2012 onwards
constant	-21.46**	-39.98**	-10.41*
	(10.63)	(20.24)	(6.12)
$\alpha(60)$	-14.82	15.46	-14.98
	(14.63)	(26.97)	(10.63)
$\alpha(120)$	-42.43***	12.05	-46.17***
	(13.89)	(24.13)	(11.83)
$\alpha(180)$	-74.25***	-38.09	-86.69***
	(18.41)	(43.40)	(17.05)
$\alpha(360)$	-70.33***	21.27	-123.70***
	(19.28)	(31.62)	(22.11)
$\lambda(36)$	8.01	11.06	5.73
	(5.28)	(8.55)	(3.65)
$\lambda(60)$	12.24***	10.46	12.71**
	(4.38)	(7.40)	(5.37)
$\lambda(120)$	16.89***	10.62*	14.03***
	(3.55)	(6.31)	(4.22)
$\lambda(180)$	39.55***	28.27	44.77***
	(10.47)	(17.50)	(10.66)
$\lambda(360)$	55.63***	45.79***	59.72***
	(9.98)	(13.14)	(14.26)
Observations	1,143	337	627
R-squared	0.21	0.23	0.22

Robust standard errors in parentheses
*** p<0.01, ** p<0.05, * p<0.1

Table 3: Non-Parametric Regression

Notes: The dependent variable is the markup of the auction i on date t computed as $(\psi_{i,t} - q_{i,t})/\psi_{i,t}$ where $q_{i,t}$ is marginal price of the auction, and $\psi_{i,t}$ is the closing secondary market price on the same day. We drop observations with an issuance of less than 1 million euros. We include both competitive and non-competitive auctions. All regressions include quarterly fixed effects. The first column is the main specification corresponding to the full sample, the second column includes all issuances up to 2010, and the third column issuances after 2012. One, two or three stars denote that the coefficient is statistically different from zero at the 10, 5, or 1% confidence level, respectively. Robust standard errors in parenthesis.

Variables	(1) Whole sample	(2) Until 2010	(3) From 2012 onwards
$\alpha(36)$	-20.75***	-32.86**	-12.52*
	(7.76)	(14.89)	(7.50)
$\alpha(60)$	-13.44	10.29	-27.18***
	(8.43)	(14.50)	(9.12)
$\alpha(120)$	-46.52***	-5.39	-66.28***
	(9.56)	(15.01)	(10.84)
$\alpha(180)$	-60.48***	-33.37	-78.06***
	(13.75)	(31.15)	(14.63)
$\alpha(360)$	-65.74***	21.23	-123.30***
	(17.82)	(28.93)	(21.40)
Issuance (% Monthly GDP)	2.58	3.54	1.32
	(4.18)	(6.28)	(5.45)
Issuance*Years to maturity	1.66***	1.20**	1.78***
	(0.36)	(0.47)	(0.49)
Observations	1,143	337	734
R-squared	0.21	0.22	0.23

Robust standard errors in parentheses
*** p<0.01, ** p<0.05, * p<0.1

Table 4: Parametric Regression

Notes: The dependent variable is the markup of the auction i on date t computed as $(\psi_{i,t} - q_{i,t})/\psi_{i,t}$ where $q_{i,t}$ is marginal price of the auction, and $\psi_{i,t}$ is the closing secondary market price on the same day. We drop observations with an issuance of less than 1 million euros. We include both competitive and non-competitive auctions. All regressions include quarterly fixed effects. The first column is the main specification corresponding to the full sample, the second column includes all issuances up to 2010, and the third column issuances after 2012. One, two or three stars denote that the coefficient is statistically different from zero at the 10, 5, or 1% confidence level, respectively. Robust standard errors in parenthesis.

Variables	(1) Issuances / GDP (quart %)	(2) Average Maturity (issuances)
Constant	3.74**	109.90***
	(1.78)	(14.57)
Deficit / GDP (quart, %)	0.65***	-0.90
	(0.12)	(0.88)
Debt Due / GDP (quart, %)	0.70***	-0.40
	(0.09)	(0.79)
Short-term Rate Factor (%)	-0.34	-8.79***
	(0.30)	(2.75)
Slope Factor (%)	0.40	-6.66**
	(0.39)	(2.99)
Observations	80	66
R-squared	0.85	0.25

Standard errors in parentheses
*** p<0.01, ** p<0.05, * p<0.1

Table 5: Issuance and maturity drivers

Notes: The dependent variables are (i) total issuances over quarterly GDP and (ii) the average WAM of issuances during the quarter. The short-term rate and the slope factors are computed from the yields of Spanish debt, and were provided to us by Jens Christensen. One, two or three stars denote that the coefficient is statistically different from zero at the 10, 5, or 1% confidence level, respectively. Robust standard errors in parenthesis.

Appendix (Nor for publication)

Contents

A Equivalence between PDE and integral formulations

Valuations and prices are given by continuous-time net present value formulae. Their PDE representation is the analogue of the recursive representation in discrete time and the integral formulation is the equivalent of the sequence summations. The solutions to each PDE can be recovered easily via the method of characteristics or as an immediate application of the Feynman-Kac formula. All of the PDEs in this paper have an exact solution contained in table A.

Price (PF)	PDE	${r^*}_t\psi_t\left(\tau\right) = \delta + \frac{\partial\psi}{\partial t} - \frac{\partial\psi}{\partial\tau};\ \psi(0,t) = 1$
	Integral	$e^{-\int_t^{t+\tau} r^*(u)du} + \delta\int_t^{t+\tau} e^{-\int_t^s r^*(u)du}ds$
Valuation (PF)	PDE	$r_t v_t\left(\tau\right) = \delta + \frac{\partial v}{\partial t} - \frac{\partial v}{\partial\tau};\ v(0,t) = 1$
	Integral	$e^{-\int_t^{t+\tau} r(u)du} + \delta\int_t^{t+\tau} e^{-\int_t^s r(u)du}ds$
Price (default)	PDE	${\hat{r}^*}_t\hat{\psi}_t\left(\tau\right) = \delta + \frac{\partial\hat{\psi}}{\partial t} - \frac{\partial\hat{\psi}}{\partial\tau} + \phi\mathbb{E}_t^X\left[\Phi\left(V\left[\hat{f}\left(\cdot,t\right)\right]\right)\psi\left(\tau,t,X_t\right) - \hat{\psi}_t\left(\tau\right)\right];\ v(0,t) = 1$
	Integral	$e^{-\int_t^{t+\tau}(\hat{r}^*(u)+\phi)du} + \int_t^{t+\tau} e^{-\int_t^s(\hat{r}^*(u)+\phi)du}\left(\delta + \phi\mathbb{E}_s^X\left[\Theta\left(V\left[\hat{f}\left(\cdot,t\right)\right]\right)\psi\left(\tau - s, t+s\right)\right]\right)ds$
Valuation (default)	PDE	${\hat{r}^*}_t\hat{v}_t\left(\tau\right) = \delta + \frac{\partial\hat{v}}{\partial t} - \frac{\partial\hat{v}}{\partial\tau} + \phi\left(\mathbb{E}_s^X\left[\left(\Theta\left(V_t\right) + \Omega_t\right)\frac{U'(c_t)}{U'(\hat{c}_t)}v_t\left(\tau\right)\right] - \hat{v}_t\left(\tau\right)\right);\ \hat{v}(0,t) = 1$
	Integral	$e^{-\int_t^{t+\tau}(\hat{r}^*(s)+\phi)ds} + \int_t^{t+\tau} e^{-\int_t^s(\hat{r}^*(u)+\phi)du}\left(\delta + \phi\mathbb{E}_s^X\left[\left(\Omega(t+s) + \Theta\left(V\left[\hat{f}\left(\cdot,t+s\right)\right]\right)\right)\frac{U'(c(t+s))}{U'(\hat{c}(t+s))}v\left(\tau - s, t+s\right)\right]\right)ds$
Debt Profile	PDE	$\frac{\partial f}{\partial t} = \iota_t\left(\tau\right) + \frac{\partial f}{\partial\tau}$
	Integral	$f_t\left(\tau\right) = \int_\tau^{min\{T,\tau+t\}} \iota(s, t+\tau-s)ds + \mathbb{I}[T > t+\tau]\cdot f(0,\tau+t)$

Table 6: Equivalence between PDE and integral formulations.

B Public Finance Considerations

B.1 A public finance microfoundation

The goal of this Appendix is to recast our original problem as a problem with distorting taxation. We modify the original model and let the government maximize the utility of households who now also supply labor. Labor taxes are the only distorting taxation. Expenditures are stochastic. The household utility is now:

$$U(c_t - \frac{h_t^{1+\nu}}{1+\nu}), \text{ where } U(\cdot) \equiv \frac{x^{1-\sigma}-1}{1-\sigma}.$$

In this case, h_t stands for hours worked and c_t household consumption. For simplicity, we assume that output is linear in hours, setting the real wage to 1. The preferences are thus GHH, with χ a disutility scale parameter and ν the inverse Frisch elasticity. Households satisfy the following budget constraint:

$$c_t = (1-\eta_t) \cdot h_t,$$

where η_t is a labor tax. We assume, that the government saves on behalf of households. Also, the only possible way that the government can transfer resources to households is through tax subsidy. The problem of the household is static, and thus, given by:

$$\max_h U(c - \chi \frac{h^{1+\nu}}{1+\nu}) \text{ subject to } c = (1-\eta_t) \cdot h.$$

Lifetime utility is given by:

$$\int_0^\infty e^{-\rho t} U\left(c_t - \chi \frac{h_t^{1+\nu}}{1+\nu}\right) dt.$$

The tax receipts for the government are now given by:

$$w_t = \eta_t \cdot h_t.$$

We assume that the government faces random expenditures, g_t, that follow a Poisson process. We can assume that g_t is negative if the government has access to some endowment, for example of natural resources. The government's budget constraint is:

$$\int_0^T q_t(\iota,\tau)\iota(\tau,t)\, d\tau = w_t + g_t + \left[f_t(0) + \delta \int_0^T f_t(\tau)\, d\tau\right].$$

For convenience, note that if we define:

$$y_t = -g_t,$$

and set η_t to zero, we are back in the budget constraint in the main body of the paper. to map the current version of the model to the one encountered earlier. Observe that the difference from the original problem is that now we allow the government to potentially bear a negative value for y_t. This is possible because households can provide labor to make up for the deficit, something that is not possible in the original problem.

Definition 1. The problem of optimal maturity with distortionary labor taxes is:

$$\max_{\{\eta_t,\iota_t(\tau)\}} \int_0^\infty e^{-\rho t} U\left(c_t - \chi \frac{h_t^{1+\nu}}{1+\nu}\right) \cdot dt$$

subject to: (i) that h is chosen optimally by the household given η, (ii) the government budget constraint, and the law of motion of debt (6), with an initial condition f_0.

Below, we prove that the solution to the problem with distortionary taxes is given by the solution to a modified version of the original problem without distorting taxes, where U is replaced by a modified return function. In particular, we prove the following result:

Proposition 8. *The solution to the optimal issuances 1 are given by the solution to*

$$\max_{\{\iota_t(\tau)\}} \int_0^\infty e^{-\rho t} U(W(x_t))dt$$

subject to the budget constraint

$$x_t = -y_t + \left[f_t(0) + \delta \int_0^T f_t(\tau)\, d\tau\right] - \int_0^T q_t(\iota,\tau)\iota(\tau,t)\, d\tau,$$

and the law of motion of debt (6) with an initial condition f_0. The function W is given by:

$$W(x) \equiv \left\{c | c - \chi^{-\frac{1}{1+\nu}} \cdot c^{\frac{1}{1+\nu}} = x\right\}$$

with domain in all x such that $W(x) \geq \left(\frac{\chi^{-\frac{1}{1+\nu}}}{1+\nu}\right)^{\frac{1+\nu}{\nu}}$. Finally, given the path of x_t consistent with the solution to $\iota_t(\tau)$, the optimal consumption and labor, and taxes in 1 are given by:

$$c_t = W(x_t)$$

$$h_t = \left(\frac{W(x_t)}{\chi}\right)^{\frac{1}{1+\nu}}$$

and

$$\eta_t = 1 - \chi\left(\frac{W(x_t)}{\chi}\right)^{\frac{\nu}{1+\nu}}.$$

Proposition 8 shows that Problem 1 can be solved by first solving the problem without distortionary taxes with a modified objective function, and then backing out the optimal taxes from the optimal issuance rule. The following immediate corollary presents the solution presented in the body of the paper.

Corollary 1. *The optimal issuance rules in Problem 1 are given by:*

$$\iota_t(\tau) = \frac{1}{\bar{\lambda}} \cdot \frac{\psi_t(\tau) - v_t(\tau)}{\psi_t(\tau)}$$

with

$$r_t v_t(\tau) = \delta + \frac{\partial v}{\partial t} - \frac{\partial v}{\partial \tau},\ \tau \in (0,T] \text{ and } v_t(0) = 1,$$

where

$$r_t = \rho + \left(\underbrace{\sigma}_{\textit{intertemporal}} - \underbrace{\frac{W_t''}{W_t'} x_t}_{\textit{intratemporal}}\right)\frac{\dot{x}_t}{x_t}.$$

and

$$x_t = y_t + \left[f_t(0) + \delta \int_0^T f_t(\tau)\, d\tau\right] - \int_0^T q_t(\iota,\tau)\iota(\tau,t)\, d\tau$$

Proof of Proposition 8. The first step is to add the household's and government's budget constraints to obtain an aggregate budget constraint

$$g_t + c_t + \left[f_t\left(0\right) + \delta \int_0^T f_t\left(\tau\right) d\tau\right] = h_t + \int_0^T q_t(\iota,\tau)\iota\left(\tau,t\right) d\tau.$$

This budget equalizes absorption plus debt services to output minus and the capital account. Next, using that $g_t = -y_t$, we have that:

$$x_t \equiv y_t + \int_0^T q_t(\iota,\tau)\iota\left(\tau,t\right) d\tau - \left[f_t\left(0\right) + \delta \int_0^T f_t\left(\tau\right) d\tau\right],$$

where x_t stands for consumption minus output. That is, this variable stands for change national wealth, in other words, the current account deficit.

The second step is to solve the household's problem. The household first-order conditions are:

$$(1-\eta_t) = h_t^{\nu}$$

and recall that the household's budget constraint is:

$$c_t = (1-\eta_t)\, h_t.$$

Combining the two equations above we obtain the following relations:

$$c_t = (1-\eta_t)\, h_t = h_t^{\nu+1}.$$

Thus, in equilibrium, the term inside the household's utility can be expressed solely in terms of consumption. To see this, note that:

$$c_t - \frac{h_t^{1+\nu}}{1+\nu} = c_t - \frac{c_t}{1+\nu} = \frac{\nu}{1+\nu}\cdot c_t.$$

Therefore, we have that the immediate household utility is given by:

$$U\left(c_t - \chi\frac{h_t^{1+\nu}}{1+\nu}\right) = \left(\frac{\nu}{1+\nu}\right)^{1-\sigma} U\left(c_t\right).$$

We can ignore the scale when we move back to solving the objective function.

The third step is to map x_t to a value for c_t. For that, we observe that:

$$c^{\frac{1}{1+\nu}} = h$$

thus, we have that:

$$x = \Gamma\left(c\right) \equiv c - c^{\frac{1}{1+\nu}}.$$

The interpretation of $\Gamma\left(c\right)$ is that it maps a level of consumption together with an equilibrium labor market choice, to a level of the current account deficit.

Shape of Γ. Next, we investigate the shape of $\Gamma\left(c\right)$.

This function satisfies:

$$\Gamma'\left(c\right) = 1 - \frac{c^{\frac{-\nu}{1+\nu}}}{1+\nu} \text{ and } \Gamma''(c) = \nu\frac{c^{\frac{-1-2\nu}{1+\nu}}}{(1+\nu)^2} > 0.$$

Thus, it is a convex function. Therefore, it has a unique minimum which is achieved at:

$$\bar{c} = \left(\frac{1}{1+\nu}\right)^{\frac{1+\nu}{\nu}}.$$

and has roots at $c = \{0, 1\}$. Hence, the function is increasing in x_t in the region $c > \bar{c}$. Then, we can obtain the maximum value of Γ:

$$\Gamma\left(\bar{c}\right) = \left(\frac{1}{1+\nu}\right)^{\frac{1+\nu}{\nu}} - \left(\left(\frac{1}{1+\nu}\right)^{\frac{1+\nu}{\nu}}\right)^{\frac{1}{1+\nu}}$$

and thus, obtain:

$$\bar{x} \equiv \Gamma\left(\bar{c}\right) = \left(\frac{1}{1+\nu}\right)^{\frac{1+\nu}{\nu}} - \left(\frac{1}{1+\nu}\right)^{\frac{1}{\nu}} = -\left(\frac{1}{1+\nu}\right)^{\frac{1}{\nu}}\left(\frac{\nu}{1+\nu}\right) \leq 0.$$

Past, $\bar{c}$ the function is decreasing. Thus, for any $c \geq \bar{c}$ we can define the inverse:

$$W\left(x\right) = \Gamma^{-1}\left(x\right) \text{ for x} \geq \bar{x}.$$

The inverse is increasing in the region Now, observe that for any $x \geq \bar{x}$ we can map x to a value of consumption.

Next, observe that if the government ever reaches a point where $x<\bar{x}$ the government cannot induce a higher current account surplus. To induce a higher current account surplus, households need to work more, but to do so, they need a greater wage subsidy. The issue is that past that subsidy, the leisure disutility income effect is so large that it induces more consumption. Thus, $\bar{x}$ is as a satiation point for the government. Thus, an optimum solution to the government problem will restrict the solution such that $x_t \geq \bar{x}$ at all t. Thus, $\bar{x}$ is the peak of a Laffer curve in this model.

Next, observe that we have

$$U\left(c_t - \chi\frac{h_t^{1+\nu}}{1+\nu}\right) = \left(\frac{\nu}{1+\nu}\right)^{1-\sigma} U\left(c_t\right) = \left(\frac{\nu}{1+\nu}\right)^{1-\sigma} U\left(W\left(x_t\right)\right)$$

when the labor market is at equilibrium, for $x_t \geq \bar{x}$. Thus, the objective of the government in the modified problem is:

$$V\left(f_0\right) = \max_{\{\iota_t(\tau)\}} \left(\frac{\nu}{1+\nu}\right)^{1-\sigma} \int_0^\infty e^{-\rho t} U\left(U\left(W\left(x_t\right)\right)\right) \cdot dt$$

where

$$x_t = y_t + \int_0^T q_t(\iota,\tau)\iota\left(\tau,t\right)d\tau - \left[f_t\left(0\right) + \delta\int_0^T f_t\left(\tau\right)d\tau\right].$$

with the restriction that $x_t \geq \underline{\text{x}}$. The problem is identical to the original version of the problem without labor taxes. Thus, their solutions must coincide.

Optimal Issuances. Define now:

$$Z\left(x\right) = U\left(W\left(x\right)\right).$$

Thus, Z is the indirect utility associated with a current account deficit. Next, note that:

$$r_t = \rho - \frac{\left(Z''\cdot\right)}{Z'}x \cdot \frac{\dot{x}}{x},$$

as we showed in the body of the text. Then, we have that:

$$Z' = U'W' \text{ and Z'' } = U'' \cdot W' + U'W''.$$

Therefore:

$$r_t = \rho - \frac{(U'' \cdot W' + U'W'')}{U'W'} x \cdot \frac{\dot{x}}{x} = \rho - \frac{U''}{U'} W \frac{x}{W} \frac{\dot{x}}{x} - \frac{W''}{W'} W \frac{1}{W} \frac{\dot{x}}{x}$$

and note that:

$$\sigma = -\frac{U''}{U'} W.$$

Thus, we have that:

$$r_t = \rho + \left(\sigma \frac{x}{W} - \frac{W''}{W'} \right) \frac{\dot{x}}{x}.$$

The rest of the formulas are as before:

$$\iota_t(\tau) = \frac{1}{\bar{\lambda}} . \frac{\psi_t(\tau) - v_t(\tau)}{\psi_t(\tau)}$$

and

$$r_t v_t(\tau) = \delta + \frac{\partial v}{\partial t} - \frac{\partial v}{\partial \tau}, \ \tau \in (0, T] \text{ and } v_t(0) = 1.$$

Finally, we use:

$$c_t = W(x_t), \ h_t = (c_t)^{\frac{1}{1+\nu}} = W(x_t)^{\frac{1}{1+\nu}} \text{ and } \eta_t = 1 - W(x_t).$$

C Proofs

C.1 Microfoundation of liquidity costs

Solution. We now provide a first order linear approximation for the price at the auction, $q_t(\iota, \tau)$, for small issuances. The result is given by the following proposition:

Proposition 9. *A first-order Taylor expansion around $\iota = 0$ yields a linear auction price:*

$$q_t(\iota, \tau) \approx \psi_t(\tau) - \frac{1}{2}\frac{\eta}{\mu y_{ss}}\psi_t(\tau)\iota_t(\tau). \tag{29}$$

Thus, the approximate liquidity cost function is $\lambda_t(\tau, \iota) \approx \frac{1}{2}\bar{\lambda}\psi_t(\tau)\iota_t(\tau)$ where the price impact is given by $\bar{\lambda}$=$\frac{\eta}{\mu y_{ss}}$.

We analyze a bond issued at time t with maturity τ. At time $t+s$, after a period of time s has passed since the auction, the time to maturity is $\tau' = \tau - s$. The valuation of the bond by investors in the secondary market is defined as:

$$\psi^{(t,\tau)}(\tau', s) \equiv \psi_{t+s}(\tau - s).$$

Hence, the price equation satisfies the PDE (2):

$$r^*_{t+s}\psi^{(t,\tau)}(\tau', s) = \delta - \frac{\partial \psi^{(t,\tau)}}{\partial \tau'} + \frac{\partial \psi^{(t,\tau)}}{\partial t},$$

with the terminal condition of $\psi^{(t,\tau)}(0, s) = 1$.

The valuation of the cash flows of the bond from the perspective of the primary dealer is $q^{(t,\tau)}(\tau', s)$. Dealers are risk neutral but have a higher cost of capital. At each moment $t+s$ dealers meet investors and sell at a price $\psi^{(t,\tau)}(\tau', s)$. The valuation of the dealers, $q^{(t,\tau)}(\tau', s)$ satisfies:

$$(r^*_{t+s} + \eta)q^{(t,\tau)}(\tau', s) = \delta - \frac{\partial q^{(t,\tau)}}{\partial \tau'} + \frac{\partial q^{(t,\tau)}}{\partial t} + \gamma^{(t,\tau)}(s)\left(\psi^{(t,\tau)}(\tau', s) - q^{(t,\tau)}(\tau', s)\right). \tag{30}$$

This expression takes this form because the dealer extracts surplus $\left(\psi^{(t,\tau)}(\tau', s) - q^{(t,\tau)}(\tau', s)\right)$ when he is matched to an investor. Before a match, primary dealers earn the flow utility, but upon a match, their value jumps to $\psi^{(t,\tau)} - q^{(t,\tau)}$. This jump arrives with endogenous intensity $\gamma^{(t,\tau)}(s)$. The complication with this PDE is its terminal condition. If $\bar{s} \le \tau$, the PDE's terminal condition is given by $q^{(t,\tau)}(\tau', \bar{s}) = \psi^{(t,\tau)}(\tau', \bar{s})$. If $\bar{s} > \tau$, the corresponding terminal condition is $q^{(t,\tau)}(0, s) = 1$, since by the expiration date, the investor is paid the principal equal to 1.

On the date of the auction, $s = 0$, $\tau' = \tau$, the dealer pays his expected bond valuation, hence the bond price demand faced by the government is:

$$q_t(\iota, \tau) \equiv q^{(t,\tau)}(\tau, 0).$$

This is because banks have free entry into the auction.

Proof. *Step 1. Exact solutions.* The solution to $q_t(\iota, \tau)$ falls into one of two cases. Case 1. If $\bar{s} \le \tau$, then:

$$q_t(\iota, \tau) = \frac{\int_0^{\bar{s}} e^{-\int_0^v \left(r^*_{t+u} + \eta\right) du} \left(\delta(\bar{s} - v) + \psi_{t+v}(\tau - v)\right) dv}{\bar{s}}. \tag{31}$$

Case 2. If $\bar{s} > \tau$, then:

$$\begin{aligned} q_t(\iota,\tau) &= \int_0^{\tau} e^{-\int_0^s \left(r^*_{t+u}+\eta\right)du} \left(\frac{\delta(\bar{s}-v)+\psi_{t+v}(\tau-v)}{\bar{s}}\right) dv \\ &\quad + e^{-\int_0^{\tau}\left(r^*_{t+u}+\eta\right)du}\frac{(\bar{s}-\tau)}{\bar{s}}. \end{aligned} \tag{32}$$

We solve the PDE for q depending on the corresponding terminal conditions, $q^{(t,\tau)}(\tau',\bar{s}) = \psi^{(t,\tau)}(\tau',\bar{s})$ and $q^{(t,\tau)}(0,s) = 1$.

Case 1. Consider the first case. The general solution to the PDE equation for $q^{(t,\tau)}(\tau',s)$ is,

$$\begin{aligned} &\int_0^{\bar{s}-s} e^{-\int_0^v \left(r^*_{t+u}+\eta+\gamma_u\right)du} \left(\delta+\gamma_{s+v}\psi_{t+v}(\tau-v)\right)dv \quad + \\ &\qquad e^{-\int_0^{\bar{s}-s}\left(r^*_{t+u}+\eta+\gamma_u\right)du}\psi_{t+(\bar{s}-s)}(\tau'-(\bar{s}-s)). \end{aligned} \tag{33}$$

This can be checked by taking partial derivatives with respect to time and maturity and applying Leibniz's rule.[51] Consider the exponentials that appear in both terms of equation (33). These can be decomposed into $e^{-\int_0^v \left[r^*_{t+u}+\eta\right]du} e^{-\int_0^v \gamma_u du}$. Then, by definition of γ we have:

$$\begin{aligned} e^{-\int_0^v \gamma_u du} &= e^{-\int_0^v \frac{1}{\bar{s}-u}du} \\ &= \frac{(\bar{s}-v)}{\bar{s}}. \end{aligned} \tag{34}$$

Thus, using (34) in (33) we can re-express it as:

$$\begin{aligned} q^{(t,\tau)}(\tau',s) &= \int_0^{\bar{s}-s} e^{-\int_0^v \left(r^*_{t+u}+\eta\right)du}\frac{(\bar{s}-v)}{\bar{s}}\left(\delta+\gamma_{s+v}\psi_{t+v}(\tau-v)\right)dv \\ &\quad + e^{-\int_0^{\bar{s}-s}\left(r^*_{t+u}+\eta\right)du}\frac{s}{\bar{s}}\psi_{t+\bar{s}}(\tau'-(\bar{s}-s)). \end{aligned}$$

When we evaluate this expression at $s = 0$, $\tau' = \tau$, and we replace $\gamma(v) = \frac{1}{\bar{s}-v}$, we arrive at:

$$\begin{aligned} q_t(\iota,\tau) &\equiv q^{(t,\tau)}(\tau,0) \\ &= \int_0^{\bar{s}} e^{-\int_0^v \left(r^*_{t+u}+\eta\right)du}\left(\frac{(\bar{s}-v)}{\bar{s}}\delta + \frac{\psi_{t+v}(\tau-v)}{\bar{s}}\right)dv. \end{aligned}$$

Case 2. The proof in the second case runs parallel to Case 1 above. The general solution to the PDE in this case is:

$$\begin{aligned} q^{(t,\tau)}(\tau',s) &= \int_0^{\tau'} e^{-\int_0^v \left(r^*_{t+u}+\eta+\gamma_u\right)du}\left(\delta+\gamma_{s+v}\psi_{t+v}(\tau-v)\right)dv \\ &\quad + e^{-\int_0^v \left(r^*_{t+u}+\eta+\gamma_u\right)du}. \end{aligned}$$

When we evaluate this expression at $s = 0$, $\tau' = \tau$:

$$\begin{aligned} q_t(\iota,\tau) &= \int_0^{\tau} e^{-\int_0^s \left(r^*_{t+u}+\eta\right)du}\frac{(\bar{s}-v)}{\bar{s}}\left(\delta+\frac{\psi_{t+v}(\tau-v)}{(\bar{s}-v)}\right)dv \\ &\quad + e^{-\int_0^{\tau}\left(r^*_{t+u}+\eta\right)du}\frac{(\bar{s}-\tau)}{\bar{s}}. \end{aligned}$$

Step 2. Limit Behavior of $q_t(\iota,\tau)$. Price with zero issuances. Consider the limit $\iota_t(\tau) \longrightarrow 0$ for any $\tau > 0$, which implies that

[51] Notice that we have directly replaced the value $\psi^{(t,\tau)}(\tau',s) = \psi_{t+s}(\tau-s)$.

$\bar{s} \longrightarrow 0$. For both Case 1 and Case 2, equations (31) and (32),[52] it holds that:

$$\lim_{\iota_t(\tau)\longrightarrow 0} q_t(\iota,\tau) = \lim_{\bar{s}\longrightarrow 0} \frac{\int_0^{\bar{s}} e^{-\int_0^s \left(r^*_{t+u}+\eta\right)du} \left(\delta(\bar{s}-s)+\psi_{t+s}(\tau-s)\right) ds}{\bar{s}}.$$

Now, both the numerator and the denominator converge to zero as we take the limits. Hence, by L'Hôpital's rule, the limit of the price is the limit of the ratio of derivatives. The derivative of the numerator is obtained via Leibniz's rule and thus,

$$\begin{aligned}
\lim_{\iota_t(\tau)\longrightarrow 0} q_t(\iota,\tau) &= \lim_{\bar{s}\longrightarrow 0} \frac{\left[e^{-\int_0^s \left(r^*_{t+u}+\eta\right)du}\left(\delta(\bar{s}-s)+\psi_{t+s}(\tau-s)\right)\right] |_{s=\bar{s}}}{1} \\
&= \lim_{\bar{s}\longrightarrow 0} e^{-\int_0^{\bar{s}} \left(r^*_{t+u}+\eta\right)du}\psi_{t+\bar{s}}(\tau-\bar{s}) \\
&= \psi_t(\tau).
\end{aligned}$$

Step 3. Linear approximation of $q_t(\iota,\tau)$. The first order approximation of the function $q_t(\iota,\tau)$, the price at the auction, around $\iota = 0$ is given by:

$$q_t(\iota,\tau) \simeq q_t(\iota,\tau) |_{\iota=0} + \frac{\partial q_t(\iota,\tau)}{\partial \iota} |_{\iota=0} \; \iota_t(\tau).$$

We computed the first term in step 2. It is given by $\psi_t(\tau)$. Thus, our objective will be to obtain $\frac{\partial q_t(\iota,\tau)}{\partial \iota} |_{\iota=0}$. Observe that by definition of $\bar{s}$, it holds that:

$$\begin{aligned}
\frac{\partial q_t(\iota,\tau)}{\partial \iota} &= \frac{\partial \bar{s}}{\partial \iota}\frac{\partial q_t(\iota,\tau)}{\partial \bar{s}} \\
&= \frac{1}{\mu y_{ss}}\frac{\partial q_t(\iota,\tau)}{\partial \bar{s}},
\end{aligned}$$

where we have applied the fact that $\bar{s} = \frac{\iota_t(\tau)}{\mu y_{ss}}$. For further reference, note that

$$\frac{\partial q_t(\iota,\tau)}{\partial \iota} |_{\iota=0} = \lim_{\bar{s}\rightarrow 0} \frac{\partial q_t(\iota,\tau)}{\partial \bar{s}} \frac{1}{\mu y_{ss}}. \tag{35}$$

Step 3.1. Derivative $\frac{\partial q_t(\iota,\tau)}{\partial \bar{s}}$. Consider the price function corresponding to Case 1. The derivative of the price function with respect to $\bar{s}$ is given by:

$$\begin{aligned}
\frac{\partial q_t(\iota,\tau)}{\partial \bar{s}} &= \frac{\partial}{\partial \bar{s}}\left(\frac{\int_0^{\bar{s}} e^{-\int_0^s \left(r^*_{t+u}+\eta\right)du} \left(\delta(\bar{s}-s)+\psi_{t+s}(\tau-s)\right) ds}{\bar{s}}\right) \\
&= \frac{e^{-\int_0^{\bar{s}} (r^*_{t+u}+\eta)du}\psi(\tau-\bar{s},t+\bar{s}) + \int_0^{\bar{s}} \delta e^{-\int_0^s \left(r^*_{t+u}+\eta\right)du} ds}{\bar{s}} \\
&\quad - \frac{\int_0^{\bar{s}} e^{-\int_0^s \left(r^*_{t+u}+\eta\right)du} \left(\delta(\bar{s}-s)+\psi_{t+s}(\tau-s)\right) ds}{\bar{s}^2} \\
&= \frac{e^{-\int_0^{\bar{s}} (r^*_{t+u}+\eta)du}\psi_{t+\bar{s}}(\tau-\bar{s}) + \int_0^{\bar{s}} \delta e^{-\int_0^s \left(r^*_{t+u}+\eta\right)du} ds - q_t(\iota,\tau)}{\bar{s}}.
\end{aligned} \tag{36}$$

Note that in the last line we used the definition of $q_t(\iota,\tau)$ as given for Case 1.

Step 3.2. Re-writing the limit of $\frac{\partial q_t(\iota,\tau)}{\partial \bar{s}}$. To obtain $\frac{\partial q_t(\iota,\tau)}{\partial \iota} |_{\iota=0}$ we compute $\lim_{\bar{s}\rightarrow 0} \frac{\partial q_t(\iota,\tau)}{\partial \bar{s}}$ using equation (36). In equation (36)

[52] For every $\tau < \bar{s}$, i.e. in Case 2, it will be analogous since we are taking the limit when $\bar{s}$ converges to zero.

both the numerator and denominator converge to zero as $\bar{s} \longrightarrow 0$.[53] Thus, we employ L'Hôpital's rule to obtain the derivative of interest. The derivative of the denominator is 1. Thus, the limit of (36) is now given by:

$$\lim_{\bar{s}\longrightarrow 0} \frac{\partial q_t(\iota,\tau)}{\partial \bar{s}} = \lim_{\bar{s}\longrightarrow 0} \frac{\partial}{\partial \bar{s}} \left[e^{-\int_0^{\bar{s}} (r^*_{t+u}+\eta)du} \psi_{t+\bar{s}}(\tau-\bar{s}) + \int_0^{\bar{s}} \delta e^{-\int_0^{s} \left(r^*_{t+u}+\eta\right)du} ds - q_t(\iota,\tau) \right]. \tag{37}$$

Step 3.3. Consider the first two terms of (37). Applying Leibniz's rule:

$$\lim_{\bar{s}\longrightarrow 0} \left\{ \left(-\frac{\partial}{\partial \tau}\psi_{t+\bar{s}}(\tau-\bar{s}) + \frac{\partial}{\partial t}\psi_{t+\bar{s}}(\tau-\bar{s}) - (r^*_{t+\bar{s}}+\eta)\psi_{t+\bar{s}}(\tau-\bar{s}) \right) e^{-\int_0^{\bar{s}} \left(r^*_{t+u}+\eta\right)du} + \delta e^{-\int_0^{\bar{s}} \left(r^*_{t+u}+\eta\right)du} \right\}.$$

The previous limit is given by:

$$-\frac{\partial}{\partial \tau}\psi_t(\tau) + \frac{\partial}{\partial t}\psi_t(\tau) - ({r^*}_t+\eta)\psi_t(\tau) + \delta.$$

Using the valuation of the international investors, we can rewrite the previous equation as:

$$\begin{aligned} -\frac{\partial}{\partial \tau}\psi_t(\tau) + \frac{\partial}{\partial t}\psi_t(\tau) - ({r^*}_t+\eta)\psi_t(\tau) + \delta &= {r^*}_t\psi_t(\tau) - ({r^*}_t+\eta)\psi_t(\tau) \\ &= -\eta\psi_t(\tau). \end{aligned} \tag{38}$$

This the first two terms of the limit of $\frac{\partial q_t(\iota,\tau)}{\partial \bar{s}}$ are equal to $-\eta\psi_t(\tau)$. Computing the limit of $\frac{\partial q_t(\iota,\tau)}{\partial \bar{s}}$: last term. The last term of (37) is given by

$$\begin{aligned} -\lim_{\bar{s}\longrightarrow 0} \frac{\partial q_t(\iota,\tau)}{\partial \bar{s}} &= -\lim_{\bar{s}\longrightarrow 0} \frac{\partial q_t(\iota,\tau)}{\partial \iota}\frac{\partial \iota}{\partial \bar{s}} \\ &= -\frac{\partial q_t(\iota,\tau)}{\partial \iota} \mid_{\iota=0} \mu y_{ss}, \end{aligned} \tag{39}$$

where we used (35). Thus, from (38) and (39), the derivative (36) is given by:

$$\lim_{\bar{s}\longrightarrow 0} \frac{\partial q_t(\iota,\tau)}{\partial \bar{s}} = -\frac{\partial q_t(\iota,\tau)}{\partial \iota} \mid_{\iota=0} \mu y_{ss} - \eta\psi_t(\tau). \tag{40}$$

Plugging (40) in (35) we obtain that:

$$\frac{\partial q_t(\iota,\tau)}{\partial \iota}\Big|_{\iota=0} = \left(-\mu y_{ss}\frac{\partial q_t(\iota,\tau)}{\partial \overline{\iota}}\Big|_{\iota=0} - \eta\psi_t(\tau) \right) \frac{1}{\mu y_{ss}}.$$

Rearranging terms, we conclude that:

$$\frac{\partial q_t(\iota,\tau)}{\partial \iota}\Big|_{\iota=0} = -\frac{\eta\psi_t(\tau)}{2\mu y_{ss}}. \tag{41}$$

[53] The limits of the three terms in the numerator of equation (36) are respectively:

$$\begin{aligned} \lim_{\bar{s}\longrightarrow 0} \int_0^{\bar{s}} e^{-\int_0^{s}(r(t+u)+\eta)du} ds &= 0, \\ \lim_{\bar{s}\longrightarrow 0} e^{-\int_0^{\bar{s}}(r(t+u)+\eta)du} \psi(\tau-\bar{s}, t+\bar{s})) &= \psi_t(\tau), \\ \lim_{\bar{s}\longrightarrow 0} q_t(\iota,\tau) &= \psi_t(\tau). \end{aligned}$$

Step 4. Taylor expansion. A first-order Taylor expansion around zero emissions yields:

$$\begin{aligned} q_t(\iota,\tau) &\simeq q_t(\iota,\tau)\mid_{\iota=0} + \frac{\partial q_t(\iota,\tau)}{\partial \iota}\mid_{\iota=0} \iota_t(\tau), \\ &= \psi_t(\tau) - \frac{\eta \psi_t(\tau)}{2\mu y_{ss}} \iota_t(\tau). \end{aligned}$$

where we used (41). We can define price impact as $\bar{\lambda} = \frac{\eta}{\mu y_{ss}}$. This concludes the proof. ☐

C.2 Proof of Proposition 1

Proof. First we construct a Lagrangian on the space of functions g such that are Lebesgue integrable, $\left\| e^{-\rho t/2} g_t(\tau) \right\|^2 < \infty$. The Lagrangian, after replacing c_t from the budget constraint, is:

$$\begin{aligned} \mathcal{L}[\iota, f] &= \int_0^\infty e^{-\rho t} U\left(y_t - f_t(0) + \int_0^T \left[q(\tau,t,\iota)\,\iota_t(\tau) - \delta f_t(\tau) \right] d\tau \right) dt \\ &\quad + \int_0^\infty \int_0^T e^{-\rho t} j_t(\tau) \left(-\frac{\partial f}{\partial t} + \iota_t(\tau) + \frac{\partial f}{\partial \tau} \right) d\tau dt, \end{aligned}$$

where $j_t(\tau)$ is the Lagrange multiplier associated to the law of motion of debt.

We consider a perturbation $h_t(\tau)$, $e^{-\rho t} h \in L^2([0,T] \times [0,\infty))$, around the optimal solution. Since the initial distribution f_0 is given, any feasible perturbation must satisfy $h_0(\tau) = 0$. In addition, we know that $f_t(T) = 0$ because $f_t(T^+) = 0$ (by construction) and issuances are infinitesimal. Thus, any admissible variation must also feature $h_t(T) = 0$. At an optimal solution f, the Lagrangian must satisfy$\mathcal{L}[\iota, f] \geq \mathcal{L}[\iota, f + \alpha h]$ for any perturbation $h_t(\tau)$.

Taking the derivative with respect to α —i.e., computing the Gâteaux derivative, for any suitable $h_t(\tau)$ we obtain:

$$\begin{aligned} \frac{d}{d\alpha} \mathcal{L}[\iota, f + \alpha h] \Big|_{\alpha=0} &= \int_0^\infty e^{-\rho t} U'(c_t) \left[-h_t(0) - \int_0^T \delta h_t(\tau)\, d\tau \right] dt \\ &\quad - \int_0^\infty \int_0^T e^{-\rho t} \frac{\partial h}{\partial t} j_t(\tau)\, d\tau dt \\ &\quad + \int_0^\infty \int_0^T e^{-\rho t} \frac{\partial h}{\partial \tau} j_t(\tau)\, d\tau dt. \end{aligned}$$

We employ integration by parts to show that:

$$\begin{aligned} \int_0^\infty \int_0^T e^{-\rho t} \frac{\partial h}{\partial t} j_t(\tau)\, d\tau dt &= \int_0^T \int_0^\infty e^{-\rho t} \frac{\partial h}{\partial t} j_t(\tau)\, dt d\tau \\ &= \int_0^T \left(\lim_{s\to\infty} e^{-\rho s} h_s(\tau)\, j_s(\tau)] - h_0(\tau)\, j_0(\tau) \right) d\tau \\ &\quad - \int_0^T \int_0^\infty e^{-\rho t} \left(\frac{\partial j_t(\tau)}{\partial t} - \rho j_t(\tau) \right) h_t(\tau)\, dt d\tau, \end{aligned}$$

and

$$\int_0^\infty e^{-\rho t} \int_0^T \frac{\partial h}{\partial \tau} j_t(\tau)\, d\tau dt = \int_0^\infty e^{-\rho t} \left[h_t(T)\, j_t(T) - h_t(0)\, j_t(0) - \int_0^T h_t(\tau) \frac{\partial j}{\partial \tau} d\tau \right] dt.$$

Replacing these calculations in the Lagrangian, and equating it to zero, yields:

$$\begin{aligned}
0 &= \int_0^\infty e^{-\rho t} U'(c_t) \left[-h_t(0) - \int_0^T \delta h_t(\tau) \, d\tau \right] dt \\
&\quad + \int_0^\infty \int_0^T e^{-\rho t} \left(-\rho j - \frac{\partial j}{\partial \tau} + \frac{\partial j}{\partial t} \right) h_t(\tau) \, d\tau dt \\
&\quad + \int_0^\infty e^{-\rho t} \left(h_t(T) j_t(T) - h_t(0) j_t(0) \right) dt \\
&\quad - \int_0^\infty \lim_{s \to \infty} e^{-\rho s} h_s(\tau) j_s(\tau) \, d\tau + h_0(\tau) j_0(\tau).
\end{aligned}$$

We rearrange terms to obtain:

$$\begin{aligned}
0 &= -\int_0^\infty e^{-\rho t} \left[U'(c_t) - j_t(0) \right] h_t(0) \, dt \\
&\quad + \int_0^\infty \int_0^T e^{-\rho t} \left(-\rho j - U'(c) \delta - \frac{\partial j}{\partial \tau} + \frac{\partial j}{\partial t} \right) h_t(\tau) \, d\tau dt \\
&\quad - \int_0^\infty e^{-\rho t} \left(h_t(T) j_t(T) \right) dt \\
&\quad - \int_0^\infty \lim_{s \to \infty} e^{-\rho s} h_s(\tau) j_s(\tau) \, d\tau + h_0(\tau) j_0(\tau).
\end{aligned} \tag{42}$$

Since $h_t(T) = h_0(\tau) = 0$ is a condition for any admissible variation, then, both the third line in equation (42) and the second term in the fourth line are equal to zero. Furthermore, because (42) needs to hold for any feasible variation $h_t(\tau)$, all the terms that multiply $h_t(\tau)$ should equal zero. The latter, yields a system of necessary conditions for the Lagrange multipliers:

$$\begin{aligned}
\rho j_t(\tau) &= -\delta U'(c_t) + \frac{\partial j}{\partial t} - \frac{\partial j}{\partial \tau}, \text{ if } \tau \in (0, T], \\
j_t(0) &= -U'(c_t), \text{ if } \tau = 0, \\
\lim_{t \to \infty} e^{-\rho t} j_t(\tau) &= 0, \text{ if } \tau \in (0, T].
\end{aligned} \tag{43}$$

Next, we perturb the control. We proceed in a similar fashion:

$$\begin{aligned}
\left. \tfrac{d}{d\alpha} \mathcal{L}[\iota + \alpha h, , f] \right|_{\alpha=0} = &\int_0^\infty e^{-\rho t} U'(c_t) \left[\int_0^T \left(\frac{\partial q}{\partial \iota} \iota_t(\tau) + q_t(\tau, \iota) \right) h_t(\tau) \, d\tau \right] dt \\
&+ \int_0^\infty \int_0^T e^{-\rho t} h_t(\tau) j_t(\tau) \, d\tau dt.
\end{aligned}$$

Collecting terms and setting the Lagrangian to zero, we obtain:

$$\int_0^\infty \int_0^T e^{-\rho t} \left[j_t(\tau) + U'(c_t) \left(\frac{\partial q}{\partial \iota} \iota_t(\tau) + q_t(\tau, \iota) \right) \right] h_t(\tau) \, d\tau dt = 0.$$

Thus, setting the term in parenthesis to zero, amounts to setting:

$$U'(c_t) \left(\frac{\partial q}{\partial \iota} \iota_t(\tau) + q_t(\tau, \iota) \right) = -j_t(\tau). \tag{44}$$

Next, we define the Lagrange multiplier in terms of goods:

$$v_t(\tau) = -j_t(\tau) / U'(c_t). \tag{45}$$

Taking the derivative of $v_t(\tau)$ with respect to t and τ we can express the necessary conditions, (43) in terms of v. In particular, we transform the PDE in (43) into the summary equations in the Proposition. That is:

$$\begin{aligned}\left(\rho-\frac{U''(c_t)\,c_t}{U'(c_t)}\frac{\dot{c}_t}{c_t}\right)v_t(\tau) &= \delta+\frac{\partial v}{\partial t}-\frac{\partial v}{\partial \tau},\ \text{if } \tau\in(0,T],\\ v_t(0) &= 1,\ \text{if } \tau=0,\\ \lim_{t\to\infty}e^{-\rho t}v_t(\tau) &= 0,\ \text{if } \tau\in(0,T];\end{aligned}$$

and the first-order condition, (44), is now given by:

$$\frac{\partial q}{\partial \iota}\iota_t(\tau)+q_t(\tau,\iota)=v_t(\tau)$$

as we intended to show. □

C.3 Proof of Lemma 1

We establish the sign relationship between ϵ_θ^μ and $\frac{\partial \epsilon_{t,\theta}^\tau}{\partial \tau}$. First, observe that if $\iota_t(s)>0$, for all $\tau\in[0,T]$

$$\mu_t=\int_0^T \tau\frac{\iota_t(\tau)}{\int_0^T\iota_t(z)\,dz}d\tau$$

is an expectation: $\mu_t=\mathbb{E}_{g_t}[\tau]$ under

$$g_t(\tau)\equiv\frac{\iota_t(\tau)}{\int_0^T\iota_t(z)\,dz},$$

a density function in $s\in[0,T]$. Let

$$G_t(\tau)\equiv\int_0^\tau g_t(s)\,ds,$$

be the cumulative distribution associated with g. We index $\{G,g\}$ by θ, a parameter of interest that affects issuances in a comparative statics. Naturally, these distributions move continuously with θ. The following argument invokes first-order stochastic dominance. Consider two arbitrary values of the parameter of interest, θ and θ' such that $\theta'>\theta$. By definition, μ_t is increasing in θ if and only if

$$\mu_t(\theta)=\mathbb{E}_{g_t(\theta)}[\tau]\leq\mathbb{E}_{g_t(\theta')}[\tau]=\mu_t(\theta').$$

Observe that since τ is increasing, then, by definition of first-order stochastic dominance, the condition above is identical to

$$G(\tau;\theta)\geq G(\tau;\theta'),$$

for all $\tau\in[0,T]$. Because ι is a continuous and bounded function of θ, this condition is equivalent to the local condition:

$$\frac{\partial}{\partial\theta}[G(\tau;\theta)]\leq 0,\ \forall\tau\in[0,T].$$

Next, we translate the conditions on G into a condition related to the elasticity of issuances. Observe that

$$\frac{\partial}{\partial\theta}[G(\tau;\theta)]=G(\tau;\theta)\left[\frac{\int_0^\tau\frac{\partial}{\partial\theta}[\iota_t(s)]\,ds}{\int_0^\tau\iota_t(s)\,ds}-\frac{\int_0^T\frac{\partial}{\partial\theta}[\iota_t(s)]\,ds}{\int_0^T\iota_t(s)\,ds}\right].$$

Thus, since the term outside the bracket is positive, the sign of $\frac{\partial}{\partial\theta}\left[G\left(\tau;\theta\right)\right]$ depends on the sign of the term inside the bracket. Thus, $\frac{\partial}{\partial\theta}\left[G\left(\tau;\theta\right)\right]\leq 0$ is equivalent to:

$$\frac{\int_0^\tau \frac{\partial}{\partial\theta}\left[\iota_t\left(s\right)\right]ds}{\int_0^\tau \iota_t\left(s\right)ds}\leq\frac{\int_0^T \frac{\partial}{\partial\theta}\left[\iota_t\left(s\right)\right]ds}{\int_0^T \iota_t\left(s\right)ds},\ \forall\tau\in\left[0,T\right].$$

To aid the calculations, we define the auxiliary function

$$H_t\left(\tau\right)\equiv\frac{\int_0^\tau \frac{\partial}{\partial\theta}\left[\iota_t\left(s\right)\right]ds}{\int_0^\tau \iota_t\left(s\right)ds},$$

and express the condition as:

$$H_t\left(\tau\right)\leq H_t\left(T\right)\ \forall\tau\in[0,T].$$

This is a necessary and sufficient condition for monotone comparative statics about the WAM.

Next, we obtain weaker stronger sufficient condition. Taking the derivative:

$$\begin{aligned}
\frac{\partial}{\partial\tau}H_t\left(\tau\right) &= H_t\left(\tau\right)\left[\frac{\iota_{t,\theta}\left(\tau\right)}{\int_0^\tau \iota_{t,\theta}\left(s\right)ds}-\frac{\iota_t\left(\tau\right)}{\int_0^\tau \iota_t\left(s\right)ds}\right]\\
&= \frac{\iota_t\left(\tau\right)}{\theta}\frac{H_t\left(\tau\right)}{\int_0^\tau \iota_{t,\theta}\left(s\right)ds}\left[\frac{\iota_{t,\theta}\left(\tau\right)}{\iota_t\left(\tau\right)}\theta-\frac{\int_0^\tau \iota_{t,\theta}\left(s\right)ds}{\int_0^\tau \iota_t\left(s\right)ds}\theta\right]\\
&= \frac{\iota_t\left(\tau\right)}{\theta}\frac{\frac{\int_0^\tau \iota_{t,\theta}(s)ds}{\int_0^\tau \iota_t(s)ds}}{\int_0^\tau \iota_{t,\theta}\left(s\right)ds}\left[\frac{\iota_{t,\theta}\left(\tau\right)}{\iota_t\left(\tau\right)}\theta-\frac{\int_0^\tau \iota_{t,\theta}\left(s\right)ds}{\int_0^\tau \iota_t\left(s\right)ds}\theta\right]\\
&= \frac{\iota_t\left(\tau\right)}{\int_0^\tau \theta\iota_t\left(s\right)ds}\left[\frac{\iota_{t,\theta}\left(\tau\right)}{\iota_t\left(\tau\right)}\theta-\frac{\int_0^\tau \iota_{t,\theta}\left(s\right)ds}{\int_0^\tau \iota_t\left(s\right)ds}\theta\right].\\
&= \frac{\iota_t\left(\tau\right)}{\int_0^\tau \theta\iota_t\left(s\right)ds}\left[\frac{\iota_{t,\theta}\left(\tau\right)}{\iota_t\left(\tau\right)}\theta-\int_0^\tau \frac{\iota_{t,\theta}\left(s\right)\theta}{\iota_t\left(s\right)}\frac{\iota_t\left(s\right)}{\int_0^\tau \iota_t\left(z\right)dz}ds\right]\\
&= \frac{\iota_t\left(\tau\right)}{\int_0^\tau \theta\iota_t\left(s\right)ds}\left[\epsilon_{t,\theta}^\tau-\mathbb{E}_{g_t(\theta')}\left[\epsilon_{t,\theta}^s|s<\tau\right]\right],
\end{aligned}$$

where $\iota_{t,\theta}\left(\tau\right)\equiv\frac{\partial\iota_t(\tau)}{\partial\theta}$. The term outside the bracket is positive by assumption. Thus,

$$sign\left(\frac{\partial}{\partial\tau}H_t\left(\tau\right)\right)=sign\left(\epsilon_{t,\theta}^\tau-\mathbb{E}_{g_t(\theta')}\left[\epsilon_{t,\theta}^s|s<\tau\right]\right).$$

If $\epsilon_{t,\theta}^\tau$ is increasing in τthen $\epsilon_{t,\theta}^\tau-\mathbb{E}_{g_t(\theta')}\left[\epsilon_{t,\theta}^s|s<\tau\right]>0$, and hence $\frac{\partial}{\partial\tau}H_t\left(\tau\right)>0$ and μ_t increases with θ. The reverse result applies if $\epsilon_{t,\theta}^\tau$ is decreasing.

C.4 Duality

Given a path of resources y_t, the primal problem is given by:

$$V\left[f_0\left(\cdot\right)\right]=\max_{\{\iota_t(\tau),c_t\}_{t\in[0,\infty),\tau\in[0,T]}}\int_t^\infty e^{-\rho(s-t)}u(c(s))ds \text{ s.t.}$$

$$c_t=y_t-f_t\left(0\right)+\int_0^T\left[q_t\left(\tau,\iota\right)\iota_t\left(\tau\right)-\delta f_t\left(\tau\right)\right]d\tau$$

$$\frac{\partial f}{\partial t}=\iota_t\left(\tau\right)+\frac{\partial f}{\partial \tau}.$$

Here we show that this problem has a dual formulation. This dual formulation, minimizes the resources needed to sustain a given path of consumption c_t:

$$D\left[f_0\left(\cdot\right)\right]=\min_{\left\{\iota_t(\tau)\right\}_{t\in[0,\infty),\tau\in[0,T]}}\int_0^\infty e^{-\int_0^t r(s)ds}y_t dt \text{ s.t.}$$

$$c_t=y_t-f_t\left(0\right)+\int_0^T\left[q(\tau,t,\iota)\iota_t\left(\tau\right)-\delta f_t\left(\tau\right)\right]d\tau$$

$$\frac{\partial f}{\partial t}=\iota_t\left(\tau\right)+\frac{\partial f}{\partial \tau};$$

$$r_t=\rho-\frac{U''\left(c_t\right)c_t}{U'\left(c_t\right)}\frac{\dot{c}_t}{c_t}.$$

Proposition 10. *Consider the solution $\left\{c_t^*,\iota_t^*\left(\tau\right),f_t^*\left(\tau\right)\right\}_{t\geq 0,\tau\in(0,T]}$ to the Primal Problem given f_0. Then, given the path of consumption c_t^*, $\left\{y_t^*,\iota_t^*\left(\tau\right),f_t^*\left(\tau\right)\right\}_{t\in[0,\infty),\tau\in(0,T]}$ solves the Dual Problem where:*

$$y_t^*=c_t^*+f_t^*\left(0\right)+\int_0^T\left[q_t(\tau,\iota^*)\iota_t^*\left(\tau\right)-\delta f_t^*\left(\tau\right)\right]d\tau.$$

Proof. Step 1. We start following the steps of Proposition 1. We construct the Lagrangian for the Dual Problem in the space $\left\|e^{-\rho t/2}g_t\left(\tau\right)\right\|^2<\infty$. After replacing the resources y_t needed to support a path of consumption c_t the budget constraint, is:

$$\begin{aligned}\mathcal{L}\left[\iota,f\right] \quad &= \quad \int_0^\infty e^{-\int_0^t r_t ds}\left(c_t+f_t\left(0\right)-\int_0^T\left[q_t\left(\tau,\iota\right)\iota_t\left(\tau\right)-\delta f_t\left(\tau\right)\right]d\tau\right)dt\\ &\quad +\int_0^\infty\int_0^T e^{-\int_0^t r_t ds}v_t\left(\tau\right)\left(-\frac{\partial f}{\partial t}+\iota_t\left(\tau\right)+\frac{\partial f}{\partial \tau}\right)d\tau dt,\end{aligned}$$

where $v_t\left(\tau\right)$ is the Lagrange multiplier associated to the law of motion of debt. We again consider a perturbation $h_t\left(\tau\right), e^{-\rho t}h\in L^2\left([0,T]\times[0,\infty)\right)$, around the optimal solution. Recall that because f_0 is given, and $f_t\left(T\right)=0$, any feasible perturbation needs to meet: $h_0\left(\tau\right)=0$ and $h_t\left(T\right)=0$. At an optimal solution f, it must be the case that $\mathcal{L}\left[\iota,f\right]\geq\mathcal{L}\left[\iota,f+\alpha h\right]$ for any feasible perturbation $h_t\left(\tau\right)$. This implies that

$$\begin{aligned}\left.\frac{\partial}{\partial\alpha}\mathcal{L}\left[\iota,f+\alpha h\right]\right|_{\alpha=0} \quad &= \quad \int_0^\infty e^{-\int_0^t r_t ds}\left[h_t\left(0\right)+\int_0^T\delta h_t\left(\tau\right)d\tau\right]dt\\ &\quad -\int_0^\infty\int_0^T e^{-\int_0^t r_t ds}\frac{\partial h}{\partial t}v_t\left(\tau\right)d\tau dt\\ &\quad +\int_0^\infty\int_0^T e^{-\int_0^t r_t ds}\frac{\partial h}{\partial \tau}v_t\left(\tau\right)d\tau dt.\end{aligned}$$

We again employ integration by parts to show that:

$$
\begin{aligned}
\int_0^\infty \int_0^T e^{-\int_0^t r_t ds} \frac{\partial h}{\partial t} v_t(\tau)\, d\tau dt &= \int_0^T \int_0^\infty e^{-\int_0^t r_t ds} v_t(\tau) \frac{\partial h}{\partial t} dt d\tau \\
&= \int_0^T \left(\lim_{s\to\infty} e^{-\int_0^t r_t ds} h_s(\tau) v_s(\tau)] - h_0(\tau) v_0(\tau) \right) d\tau \\
&\quad - \int_0^T \int_0^\infty e^{-\int_0^t r_t ds} \left(\frac{\partial v_t(\tau)}{\partial t} - r_t v_t(\tau) \right) h_t(\tau)\, dt d\tau \\
&= \int_0^T \left(\lim_{s\to\infty} e^{-\int_0^t r_t ds} h_s(\tau) v_s(\tau) - h_0(\tau) v_0(\tau) \right) d\tau - \\
&\quad \int_0^\infty e^{-\int_0^s r(u) du} \int_0^T \left(\frac{\partial v_t(\tau)}{\partial t} - r_t v_t(\tau) \right) h_t(\tau)\, d\tau dt,
\end{aligned}
$$

and

$$
\int_0^\infty e^{-\int_0^t r_t ds} \int_0^T \frac{\partial h}{\partial \tau} v_t(\tau)\, d\tau dt = \int_0^\infty e^{-\int_0^t r_t ds} \left[h_t(T) v(T,t) - h_t(0) v_t(0) - \int_0^T h_t(\tau) \frac{\partial v}{\partial \tau} d\tau \right] dt.
$$

Replacing these calculations in the Lagrangian, and equating it to zero, yields:

$$
\begin{aligned}
0 &= \int_0^\infty e^{-\int_0^t r_t ds} \left[h_t(0) + \int_0^T \delta h_t(\tau)\, d\tau \right] dtt \\
&\quad + \int_0^\infty \int_0^T e^{-\int_0^t r_t ds} \left(-r_t v - \frac{\partial v}{\partial \tau} + \frac{\partial v}{\partial t} \right) h_t(\tau)\, d\tau dt \\
&\quad + \int_0^\infty e^{-\int_0^t r_t ds} \left(h_t(T) v(T,t) - h_t(0) v_t(0) \right) dt \\
&\quad - \int_0^\infty \lim_{s\to\infty} e^{-\int_0^s r(u) du} h_s(\tau) v_s(\tau)\, d\tau.
\end{aligned}
$$

Again, the previous equation needs to hold for any feasible variation $h_t(\tau)$, all the terms that multiply $h_t(\tau)$ should be equal to zero. The latter, yields a system of necessary conditions for the Lagrange multipliers, and substituting for the value of r_t:

$$
\begin{aligned}
\left(\rho - \frac{U''(c_t) c_t}{U'(c_t)} \frac{\dot{c}_t}{c_t} \right) v_t(\tau) &= \delta + \frac{\partial v}{\partial t} - \frac{\partial v}{\partial \tau}, \text{ if } \tau \in (0,T], \\
v_t(0) &= 1, \text{ if } \tau = 0, \\
\lim_{t\to\infty} e^{-\rho t} v_t(\tau) &= 0, \text{ if } \tau \in (0,T].
\end{aligned} \tag{46}
$$

By proceeding in a similar fashion with the control we arrive to:

$$
\left(\frac{\partial q}{\partial \iota} \iota_t(\tau) + q_t(\tau, \iota) \right) = -v_t(\tau). \tag{47}
$$

Note that system of equation (46) to (47) plus the budget constraint, the law of motion of debt, and initial debt f_0, are precisely the conditions that characterize the solution of the primal problem. □

C.5 No liquidity costs: $\bar{\lambda} = 0$

Proposition 11. *(Optimal Policy with Liquid Debt) Assume that $\lambda(\tau, t, \iota) = 0$. If a solution exists, then consumption satisfies equation (13) with ${r^*}_t = r_t$ and the initial condition $B(0) = \int_0^\infty \exp\left(-\int_0^s r^*(u)du\right)(c(s) - y(s))\, ds$. Given the optimal path of* *consumption, any solution $\iota_t(\tau)$ consistent with (6), and*

$$\dot{B}_t = r^*{}_t B_t + c_t - y_t, \text{ for } t > 0, \tag{48}$$

where

$$B_t = \int_0^T \psi_t(\tau) f_t(\tau) d\tau, \tag{49}$$

is an optimal solution.

Proof. Step 1. The first part of the proof is just a direct consequence of the first-order condition $v_t(\tau) = \psi_t(\tau)$ for bond issuance. Bond prices are given by (2) while the government valuations are given by (12). Since both equations must be equal in a bounded solution, we conclude that

$$r_t^* = r_t = \rho - \frac{U''(c_t)}{U'(c_t)} \frac{dc}{dt},$$

must describe the dynamics of consumption.

Step 2. The second part of the proof derives the law of motion of B_t. First we take the derivative with respect to time at both sides of definition (49). Recall that, from the law of motion of debt, equation (6), it holds that:

$$\iota_t(\tau) = -\frac{\partial f}{\partial t} + \frac{\partial f}{\partial \tau}.$$

To express the budget constraint in terms of f, we substitute $\iota_t(\tau)$ into the budget constraint:

$$c_t = y_t - f_t(0) + \int_0^T \left[\psi_t(\tau)\left(\frac{\partial f}{\partial t} - \frac{\partial f}{\partial \tau}\right) - \delta f_t(\tau)\right] d\tau. \tag{50}$$

We would like to rewrite equation (50). Therefore, first, we apply integration by parts to the following expression:

$$\int_0^T \psi_t(\tau) \frac{\partial f}{\partial \tau} d\tau = \psi_t(T) f_t(T) - \psi_t(0) f_t(0) - \int_0^T \frac{\partial \psi}{\partial \tau} f_t(\tau) d\tau.$$

As long as the solution is smooth, it holds that $f_t(T) = 0$. Further, recall that by construction $\psi_t(0) = 1$. Hence:

$$\int_0^T \psi_t(\tau) \frac{\partial f}{\partial \tau} d\tau = -f_t(0) - \int_0^T \frac{\partial \psi}{\partial \tau} f_t(\tau) d\tau. \tag{51}$$

Second, from the pricing equation of international investors, we know that

$$\frac{\partial \psi}{\partial \tau} = -r^*{}_t \psi_t(\tau) + \delta + \frac{\partial \psi}{\partial t}.$$

Then, we obtain:

$$\int_0^T \psi_t(\tau) \frac{\partial f}{\partial \tau} d\tau = -f_t(0) - \int_0^T \left[\delta + \psi_t(\tau) - r_t \psi_t(\tau)\right] f_t(\tau) d\tau. \tag{52}$$

We substitute (51) and (52) into (50), and thus:

$$
\begin{aligned}
c_t =& y_t - f_t(0) + \int_0^T \left[\psi_t(\tau)\frac{\partial f}{\partial t} - \delta f_t(\tau)\right] d\tau ... \\
& - \left\{ -f_t(0) - \int_0^T \left[\delta + \psi_t(\tau) - r_t \psi_t(\tau)\right] f_t(\tau)\, d\tau \right\} \\
=& y_t + \int_0^T \left[\psi_t(\tau) f_{tt}(\tau) + \psi_t(\tau) f_t(\tau)\right] d\tau - \int_0^T r^*{}_t \psi_t(\tau) f_t(\tau)\, d\tau.
\end{aligned}
$$

Rearranging terms and employing the definitions above, we obtain:

$$
\dot{B}_t = c_t - y_t + r^*{}_t B_t,
$$

as desired. □

C.6 Asymptotic behavior

Here we formally prove the limit conditions of Proposition 1. In particular, we provide a complete asymptotic characterization. The following Proposition provides a summary.

Proposition 12. *Assume that $\rho > r^*_{ss}$, there exists a steady state if and only if $\bar{\lambda} > \bar{\lambda}_o$ for some $\bar{\lambda}_o$. If instead, $\bar{\lambda} \leq \bar{\lambda}_o$, there is no steady state but consumption converges asymptotically to zero. In particular, the asymptotic behavior is:*

Case 1 (High Liquidity Costs). *For liquidity costs above the threshold value $\bar{\lambda} > \bar{\lambda}_o$, variables converge to a steady state characterized by the following system:*

$$
\begin{aligned}
\frac{\dot{c}_{ss}}{c_{ss}} &= 0 \\
r_{ss} &= 0 \qquad (53) \\
\iota_{ss}(\tau) &= \frac{\psi_{ss}(\tau) - v_{ss}(\tau)}{\bar{\lambda}\psi_{ss}(\tau)}, \qquad (54) \\
v_{ss}(\tau) &= \frac{\delta}{\rho}(1 - e^{-\rho\tau}) + e^{-\rho\tau} \qquad (55) \\
f_{ss}(\tau) &= \int_\tau^T \iota_{ss}(s) ds \qquad (56) \\
c_{ss} &= y_{ss} - f_{ss}(0) + \int_0^T \left[\psi_{ss}(\tau)\iota_{ss}(\tau) - \frac{\bar{\lambda}\psi_{ss}(\tau)}{2}\iota_{ss}(\tau)^2 - \delta f_{ss}(\tau)\right]. \qquad (57)
\end{aligned}
$$

Case 2 (Low Liquidity Costs). *For liquidity costs below the threshold value $0 < \bar{\lambda} \leq \bar{\lambda}_o$, variables converge asymptotically to:*

$$
\begin{aligned}
\lim_{s\to\infty} \frac{c_s}{c_t} &= e^{-\frac{(\rho - r_\infty(\bar{\lambda}))(s-t)}{\sigma}} \\
v_\infty(\tau, r_\infty(\bar{\lambda})) &= \frac{\delta}{r_\infty(\bar{\lambda})}(1 - e^{-r_\infty(\bar{\lambda})\tau}) + e^{-r_\infty(\bar{\lambda})\tau} \\
\iota_\infty(\tau, r_\infty(\bar{\lambda})) &= \frac{\psi_{ss}(\tau) - v_\infty(\tau, r_\infty(\bar{\lambda}))}{\bar{\lambda}\psi_{ss}(\tau)} \\
f_\infty(\tau, r_\infty(\bar{\lambda})) &= \int_\tau^T \iota_\infty(s, r_\infty(\bar{\lambda})) ds
\end{aligned}
$$

where $r_\infty(\bar{\lambda})$ satisfies $r^*_{ss} \le r_\infty(\bar{\lambda}) < \rho$ and solves:

$$\begin{aligned} c_\infty &= 0 \\ &= y_{ss} - f_\infty(0, r_\infty(\bar{\lambda})) + \int_0^T \left[\iota_\infty(\tau, r_\infty(\bar{\lambda}))\psi(\tau) - \frac{\bar{\lambda}\psi_{sst}(\tau)}{2}\iota_\infty(\tau, r_\infty(\bar{\lambda}))^2 - \delta f_\infty(\tau, r_\infty(\bar{\lambda}))\right] d\tau. \end{aligned}$$

Threshold. *The threshold $\bar{\lambda}_o$ solves $|c_{ss}|_{\bar{\lambda}=\bar{\lambda}_o} = 0$ in (57) and $\lim_{\bar{\lambda}\to\bar{\lambda}_o} r_\infty(\bar{\lambda}) = \rho$.*

Proof. Step 1. First observe that as $\bar{\lambda} \to \infty$, the optimal issuance policy (15) approaches $\iota_t(\tau) = 0$. Thus, for that limit, $c_{ss} = y > 0$ and $f_{ss}(\tau) = 0$.

Step 2. Next, consider the system in Case 1 of Proposition 12 as a guess of a solution. Note that equations (54) to (57) meet the necessary conditions of Proposition 1 as long as $r_t = \rho$. This because: $\iota_{ss}(\tau)$ meets the first order condition with respect to the control; $v_{ss}(\tau)$ solves the PDE for valuations; given $\iota_{ss}(\tau)$ and $v_{ss}(\tau)$ the stock of debt solves the KFE, thus, is given by $\int_\tau^T \iota_{ss}(s)ds$; and consumption is pinned down by the budget constraint. In addition, by construction, consumption determined in (57) does not depend on time; i.e. $\dot{c}_t = 0$ and this implies that

$$r_{ss} \equiv r_t = \rho.$$

Thus, the only thing we need to check is that there exists some $\bar{\lambda}$ finite such that consumption is positive.

Step 3. The system in equations (54) to (57) is continuous in $\bar{\lambda}$. Therefore, because $c_{ss} = y > 0$ for $\bar{\lambda} \to \infty$, there exists a value of $\bar{\lambda}$ such that the implied consumption by equations (54) to (57) is positive.

Step 4. We now prove that there is an interval where this solution holds. In particular, we will show that c_{ss} decreases as $\bar{\lambda}$ increases. Observe that, steady state internal valuations $v_{ss}(\tau)$ in (55) and bond prices $\psi(\tau)$ are independent of $\bar{\lambda}$. Steady-state debt issuance's $\iota_{ss}(\tau)$ in (54) are a monotonously decreasing function of $\bar{\lambda}$, because

$$\frac{\partial \iota_{ss}(\tau)}{\partial \bar{\lambda}} = -\frac{1}{\bar{\lambda}}\iota_{ss}(\tau) < 0,$$

and therefore the total amount of debt at each maturity $f_{ss}(\tau)$ in (56) is also decreasing with $\bar{\lambda}$, because

$$\frac{\partial f_{ss}(\tau)}{\partial \bar{\lambda}} = -\frac{1}{\bar{\lambda}} f_{ss}(\tau) < 0.$$

If we take derivatives with respect to $\bar{\lambda}$ in the budget constraint (57) we obtain:

$$\begin{aligned} \frac{\partial c_{ss}}{\partial \bar{\lambda}} &= - \frac{\partial f_{ss}(0)}{\partial \bar{\lambda}} + \int_0^T \left[\psi_{ss}(\tau)\frac{\partial \iota_{ss}(\tau)}{\partial \bar{\lambda}} - \frac{\psi_{ss}(\tau)}{2}\iota_{ss}(\tau)^2 - \bar{\lambda}\psi_{ss}(\tau)\iota_{ss}(\tau)\frac{\partial \iota_{ss}(\tau)}{\partial \bar{\lambda}} - \delta\frac{\partial f_{ss}(\tau)}{\partial \bar{\lambda}}\right] d\tau \\ &= \frac{1}{\bar{\lambda}} f_{ss}(0) - \frac{1}{\bar{\lambda}}\int_0^T \left[\psi_{ss}(\tau)\iota_{ss}(\tau) + \bar{\lambda}\frac{\psi_{ss}(\tau)}{2}\iota_{ss}(\tau)^2 - \bar{\lambda}\psi_{ss}(\tau)\iota_{ss}(\tau)^2 - \delta f_{ss}(\tau)\right] d\tau \\ &= -\frac{1}{\bar{\lambda}} c_{ss} < 0. \end{aligned}$$

Observe that $\iota_{ss}(\tau)$ can be made arbitrarily small by increasing $\bar{\lambda}$. Thus, there exists a value of $\bar{\lambda} \ge 0$ such that $c_{ss} = 0$ in the system above. We denote this value by $\bar{\lambda}_o$.

Step 5. For $\bar{\lambda} \le \bar{\lambda}_o$, if a steady state existed, it would imply $c_{ss} < 0$, outside of the range of admissible values. Therefore, there is no steady state in this case. Assume that the economy grows asymptotically at rate $g_\infty(\bar{\lambda}) \equiv \lim_{t\to\infty} \frac{1}{c_t}\frac{dc}{dt}$. If $g_\infty(\bar{\lambda}) > 0$ then consumption would grow to infinity, which violates the budget constraint. Thus, if there exists an asymptotic the growth rate, it is negative: $g_{\infty}\left(\bar{\lambda}\right) < 0$. If we define $r_{\infty}\left(\bar{\lambda}\right)$ as

$$r_{\infty}\left(\bar{\lambda}\right) \equiv \left(\rho + \sigma g\left(\bar{\lambda}\right)\right) < \rho,$$

the growth rate of the economy can be expressed as

$$g_{\infty}\left(\bar{\lambda}\right) = -\frac{\left(\rho - r_{\infty}\left(\bar{\lambda}\right)\right)}{\sigma}.$$

When this is the case, the asymptotic valuation is

$$v_{\infty}\left(\tau, r_{\infty}\left(\bar{\lambda}\right)\right) = \frac{\delta\left(1 - e^{-r_{\infty}(\bar{\lambda})\tau}\right)}{r_{\infty}\left(\bar{\lambda}\right)} + e^{-r_{\infty}(\bar{\lambda})\tau}.$$

To obtain the discount factor bounds, observe that if $v_{\infty}\left(\tau, r_{\infty}\left(\bar{\lambda}\right)\right) \leq \psi_{ss}\left(\tau\right)$ the optimal issuance is non-negative. Otherwise issuances would be negative at all maturities and the country would be an asymptotic net asset holder. This cannot be an optimal solution as this implies that consumption can be increased just by reducing the amount of foreign assets. Therefore, $r_{\infty}\left(\bar{\lambda}\right) \geq r^{*}$. Finally, by definition $r_{\infty}\left(\bar{\lambda}\right) < \rho$. □

C.7 Proof of Proposition 3

When $\delta = 0$, we have that

$$\iota_{ss}\left(\tau\right) = \frac{1}{\bar{\lambda}}\left(1 - \exp\left(-\left(\rho - r_{ss}^{*}\right)\tau\right)\right).$$

Define the spread, $\Delta \equiv \rho - r_{ss}^{*}$. We have that:

$$\frac{\partial}{\partial\Delta}\left[\frac{1}{\bar{\lambda}}\left(1 - \exp\left(-\Delta\tau\right)\right)\right] = \frac{1}{\bar{\lambda}}\left(\Delta\exp\left(-\Delta\tau\right)\right).$$

From here, we compute the elasticity with respect to relative impatience:

$$\epsilon_{ss,\Delta}^{\tau} = \Delta\frac{\exp\left(-\Delta\tau\right)}{1 - \exp\left(-\Delta\tau\right)} = \rho\frac{\Delta}{\exp\left(\Delta\tau\right) - 1} > 0.$$

The denominator is increasing in τ, so $\epsilon_{ss,\Delta}^{\tau}$ is decreasing in τ. Thus, by Lemma 1, the WAM decreases with impatience.

C.8 Proof of Proposition 4

Part 1: Smoothing. We investigate the effect on the WAM of a temporary drop in steady state income to the initial income y_0. Thus, we consider a decline in income starting from the steady state at time zero. We investigate the special limit case as liquidity costs are very large, $\bar{\lambda} \to \infty$, and bonds are zero coupon, $\delta = 0$, which renders a closed form solution. Note that in this case:

$$\lim_{\bar{\lambda}\to\infty} f_{ss}\left(\tau\right) = 0, \quad \lim_{\bar{\lambda}\to\infty} \iota_{t}\left(\tau\right) = 0,$$

and thus:

$$\lim_{\bar{\lambda}\to\infty} c_t = y_t.$$

Recall that the path of income is given by:

$$y_t = y_{ss} + \left(y_0 - y_{ss}\right)\exp\left(-\alpha t\right) \text{ and } \dot{y}_t = -\alpha\left(y_0 - y_{ss}\right)\exp\left(-\alpha t\right).$$

Now, consider a small negative initial drop in income near the steady-state, $\varepsilon = y_{ss} - y_0 \gtrsim 0$. Therefore, we have (in the limit as $\varepsilon \to 0$)

$$\begin{aligned}
\frac{\partial}{\partial \varepsilon}\left[\dot{y}_t/y_t\right]|_{\epsilon=0} &= \frac{\partial}{\partial \varepsilon}\left[\frac{\alpha\varepsilon\exp\left(-\alpha t\right)}{y_{ss}-\varepsilon\exp\left(-\alpha t\right)}\right]|_{\varepsilon=0} \\
&= \alpha\frac{\exp\left(-\alpha t\right)}{y_{ss}-\varepsilon\exp\left(-\alpha t\right)}|_{\epsilon=0} + \frac{\alpha\varepsilon\left(\exp\left(-\alpha t\right)\right)^2}{\left(y_{ss}-\varepsilon\exp\left(-\alpha t\right)\right)^2}|_{\varepsilon=0} \\
&= \alpha\frac{\exp\left(-\alpha t\right)}{y_{ss}}.
\end{aligned}$$

Because we are working with the large $\bar{\lambda}$ limit, we have:

$$\lim_{\bar{\lambda}\to\infty}\frac{\partial}{\partial \varepsilon}\left[\dot{c}_t/c_t\right]|_{\varepsilon=0} = \frac{\partial}{\partial \varepsilon}\left[\dot{y}_t/y_t\right]|_{\varepsilon=0} = \alpha\frac{\exp\left(-\alpha t\right)}{y_{ss}}.$$

From here, we can compute the impact on the domestic discount. Recall that:

$$r_t = \rho + \sigma \cdot \frac{\dot{c}_t}{c_t}.$$

Thus, we have that:

$$\lim_{\bar{\lambda}\to\infty}\frac{\partial}{\partial \varepsilon}\left[r_t\right] = \sigma\alpha\left[\frac{\exp\left(-\alpha t\right)}{y_{ss}}\right].$$

Now, recall that the optimal issuances at time zero are given by:

$$\iota_0\left(\tau\right) = \frac{1}{\bar{\lambda}}\left(1-\frac{v_0\left(\tau\right)}{\psi_0\left(\tau\right)}\right) = \frac{1}{\bar{\lambda}}\left(1-\exp\left(-\int_0^\tau\left(r_s-r_{ss}^*\right)ds\right)\right) > 0.$$

Thus, we have that:

$$\begin{aligned}
\epsilon_{0,\varepsilon}^\tau &\equiv \frac{\partial\iota_0\left(\tau\right)}{\partial\varepsilon}\cdot\frac{1}{\iota_0\left(\tau\right)} \\
&= \frac{\partial\left(1-\exp\left(-\int_0^\tau\left(r_s-r_{ss}^*\right)ds\right)\right)}{\partial\varepsilon}\cdot\frac{1}{\left(1-\exp\left(-\int_0^\tau\left(r_s-r_{ss}^*\right)ds\right)\right)} \\
&= \left(-\frac{\exp\left(-\int_0^\tau r_s-r_{ss}^*ds\right)}{1-\exp\left(-\int_0^\tau r_s-r_{ss}^*ds\right)}\right)\cdot\left(-\int_0^\tau\frac{\partial}{\partial\varepsilon}\left[r_s\right]ds\right) \\
&= \frac{\int\frac{\partial}{\partial\varepsilon}\left[r_s\right]ds}{\exp\left(\int_0^\tau r_s-r_{ss}^*ds\right)-1}\frac{1}{y_{ss}}.
\end{aligned}$$

Therefore, in this expression we have that:

$$\lim_{\bar{\lambda}\to\infty}\epsilon_{0,\varepsilon}^\tau = \frac{1}{y_{ss}}\sigma\alpha\frac{\int_0^\tau\exp\left(-\alpha s\right)ds}{\exp\left(\left(\rho-r^*\right)\tau\right)-1} = -\frac{1}{y_{ss}}\sigma\frac{\exp\left(-\alpha\cdot s\right)|_{s=0}^\tau}{\exp\left(\left(\rho-r^*\right)\tau\right)-1} = -\frac{1}{y_{ss}}\sigma\frac{\exp\left(-\alpha\tau\right)-1}{\exp\left(\left(\rho-r^*\right)\tau\right)-1}.$$

Finally, notice then that reversing signs we obtain

$$\lim_{\bar{\lambda}\to\infty}\epsilon_{0,\varepsilon}^\tau = \frac{1}{y_{ss}}\sigma\frac{1-\exp\left(-\alpha\tau\right)}{\exp\left(\left(\rho-r^*\right)\tau\right)-1} \gtrsim 0.$$

Thus, we have that issuances increase with the drop in income and scale with the IES coefficient. Next, we show that the WAM is decreasing with the perturbation. We need to show that $\epsilon^{\tau}_{0,\varepsilon}$ is decreasing in τ. Note that

$$\frac{\partial}{\partial \tau}\left[\epsilon^{\tau}_{0,\varepsilon}\right] = \epsilon^{\tau}_{0,\varepsilon}\left[\alpha\frac{\exp\left(-\alpha\tau\right)}{1-\exp\left(-\alpha\tau\right)} - \left(\rho - r^*\right)\frac{\exp\left(\left(\rho - r^*\right)\tau\right)}{\exp\left(\left(\rho - r^*\right)\tau\right)-1}\right].$$

Thus,

$$\begin{aligned} sign\left(\frac{\partial}{\partial \tau}\left[\epsilon^{\tau}_{0,\varepsilon}\right]\right) &= sign\left(\alpha\frac{1}{\exp\left(\alpha\tau\right)-1} - \left(\rho - r^*\right)\frac{1}{1-\exp\left(-\left(\rho - r^*\right)\tau\right)}\right) \\ &= sign\left(\frac{\alpha}{\rho - r^*} - \frac{\exp\left(\alpha\tau\right)-1}{1-\exp\left(-\left(\rho - r^*\right)\tau\right)}\right). \end{aligned}$$

Define,

$$h\left(\tau\right) \equiv \frac{\exp\left(\alpha\tau\right)-1}{1-\exp\left(-\left(\rho - r^*\right)\tau\right)} > 0.$$

By L'Hospital's rule, the function

$$\lim_{\tau\to 0}\frac{\exp\left(\alpha\tau\right)-1}{1-\exp\left(-\left(\rho - r^*\right)\tau\right)} = \frac{\alpha}{\rho - r^*}.$$

It suffices to show that $h\left(\tau\right)$ is increasing for $\tau > 0$We do it by contradiction. Suppose that $h\left(\tau\right)$ is decreasing or constant, and hence $h\left(\tau\right) \leq \frac{\alpha}{\rho - r^*}$. The derivative of this function is non-positive , $h'\left(\tau\right) \leq 0$. Then,

$$\begin{aligned} h'\left(\tau\right) &= \left[\frac{\alpha\exp\left(\alpha\tau\right)}{\exp\left(\alpha\tau\right)-1} - \frac{\left(\rho - r^*\right)\exp\left(-\left(\rho - r^*\right)\tau\right)}{1-\exp\left(-\left(\rho - r^*\right)\tau\right)}\right]\frac{\exp\left(\alpha\tau\right)-1}{1-\exp\left(-\left(\rho - r^*\right)\tau\right)} \\ &= h\left(\tau\right)\left[\alpha\frac{1}{1-\exp\left(-\alpha\tau\right)} - \left(\rho - r^*\right)\frac{1}{\exp\left(\left(\rho - r^*\right)\tau\right)-1}\right] \leq 0, \end{aligned}$$

or

$$\frac{\alpha}{\left(\rho - r^*\right)} \leq \frac{1-\exp\left(-\alpha\tau\right)}{\exp\left(\left(\rho - r^*\right)\tau\right)-1}.$$

This requires that:

$$\frac{\exp\left(\alpha\tau\right)-1}{1-\exp\left(-\left(\rho - r^*\right)\tau\right)} = h\left(\tau\right) \leq \frac{\alpha}{\rho - r^*} \leq \frac{1-\exp\left(-\alpha\tau\right)}{\exp\left(\left(\rho - r^*\right)\tau\right)-1} = \frac{\exp\left(-\alpha\tau\right)}{\exp\left(\left(\rho - r^*\right)\tau\right)}\frac{\exp\left(\alpha\tau\right)-1}{1-\exp\left(-\left(\rho - r^*\right)\tau\right)},$$

and simplifying

$$\exp\left(\left(\rho - r^*\right)\tau\right) \leq \exp\left(-\alpha\tau\right),$$

which is false for any $\tau > 0$. . Thus, by lemma 1, the WAM decreases, i.e.

$$\lim_{\bar{\lambda}\to\infty}\epsilon^{\mu}_{0,\epsilon} < 0.$$

Part 2: Yield riding.

Recall that the path of international rates is given by:

$$r^*_t = r^*_{ss} + \left(r^*_0 - r^*_{ss}\right)\exp\left(-\alpha t\right).$$

Now, consider a small initial increase in rates. Thus, we have: $r^*_0 = r^*_{ss} + \varepsilon$. With zero coupons,

$$\psi_0\left(\tau\right) = \exp\left(-\int_0^{\tau} r^*_s ds\right) = \exp\left(-r^*_{ss}\tau - \varepsilon\cdot\int_0^{\tau}\exp\left(-\alpha t\right)ds\right).$$

We compute the integral to obtain:

$$\begin{aligned}
\psi_0(\tau) &= \exp\left(-r_{ss}^*\tau + \frac{\varepsilon}{\alpha}\cdot\int_0^\tau(-\alpha)\cdot\exp(-\alpha\cdot t)\,ds\right) \\
&= \exp\left(-r_{ss}^*\tau + \frac{\varepsilon}{\alpha}\left(\exp(-\alpha\cdot\tau)-1\right)\right) \\
&= \exp\left(-r_{ss}^*\tau\right)\exp\left(\frac{\varepsilon}{\alpha}\left(\exp(-\alpha\cdot\tau)-1\right)\right) \\
&= \psi_{ss}(\tau)\cdot\exp\left(\frac{\varepsilon}{\alpha}\left(\exp(-\alpha\cdot\tau)-1\right)\right).
\end{aligned}$$

Next, the derivative (in the limit as $\varepsilon\to 0$) is given by:

$$\frac{\partial}{\partial\varepsilon}\left[\psi_0(\tau)\right]|_{\varepsilon=0} = \psi_{ss}(\tau)\frac{1}{\alpha}\left(\exp(-\alpha\cdot\tau)-1\right)\exp\left(\frac{\varepsilon}{\alpha}\left(\exp(-\alpha\cdot\tau)-1\right)\right)|_{\varepsilon=0} = \frac{1}{\alpha}\psi_{ss}(\tau)\left(\exp(-\alpha\cdot\tau)-1\right)<0$$

and

$$\psi_0(\tau)|_{\varepsilon=0} = \psi_{ss}(\tau).$$

Now, recall that issuances are:

$$\iota_0(\tau) = \frac{1}{\bar{\lambda}}\left(1-\frac{v_0(\tau)}{\psi_0(\tau)}\right)>0.$$

Thus, since $\sigma=0$ implies $v_0(\tau)=\exp(-\rho\tau)$, we have that:

$$\begin{aligned}
\frac{\partial}{\partial\varepsilon}\left[\iota_0(\tau)\right]|_{\epsilon=0} &= -\frac{1}{\bar{\lambda}}\left(-\frac{v_0(\tau)}{\psi_0(\tau)}\frac{\frac{\partial}{\partial\varepsilon}\left[\psi_0(\tau)\right]}{\psi_0(\tau)}\right)|_{\epsilon=0} \\
&= \frac{1}{\alpha\bar{\lambda}}\exp\left(-\left(\rho-r_{ss}^*\right)\tau\right)\left(\exp(-\alpha\cdot\tau)-1\right)).
\end{aligned}$$

Therefore,

$$\epsilon_{0,\varepsilon}^{\tau} = \frac{1}{\iota_0(\tau)}\frac{\partial}{\partial\epsilon}\left[\iota_0(\tau)\right] = \frac{1}{\alpha}\frac{\exp\left(-\left(\rho-r_{ss}^*\right)\tau\right)\left(\exp(-\alpha\cdot\tau)-1\right))}{1-\left(\exp\left(-\left(\rho-r_{ss}^*\right)\tau\right)\right)} = -\frac{1}{\alpha}\frac{1-\exp(-\alpha\cdot\tau)}{\exp\left(\left(\rho-r_{ss}^*\right)\tau\right)-1}.$$

In the proof of the effect of smoothing, we showed that the function $\frac{1-\exp(-\alpha\cdot\tau)}{\exp((\rho-r_{ss}^*)\tau)-1}$ is decreasing, then multiplying it by the constant $(-1/\alpha)$ makes $\epsilon_{0,\varepsilon}^{\tau}$ increasing in τ. Thus, the WAM increases with a temporary positive increase in the level of interest rates—and the yield curve slopes downward.

C.9 Limiting distribution: $\bar{\lambda}\to 0$

Proposition 13. *(Limiting distribution) In the limit as liquidity costs vanish, $\bar{\lambda}\to 0$, the asymptotic optimal issuance is given by*

$$\iota_\infty^{\bar{\lambda}\to 0}(\tau) = \lim_{\bar{\lambda}\to 0}\iota_\infty(\tau) = \frac{1+\left[-1+\left(r^*/\delta-1\right)r_{ss}^*\tau\right]e^{-r_{ss}^*\tau}}{1+\left[-1+\left(r^*/\delta-1\right)r_{ss}^*T\right]e^{-r_{ss}^*T}}\frac{\psi_{sst}}{\psi_{ss}(\tau)}\kappa, \tag{58}$$

where constant $\kappa>0$ is such that $y_{ss}-f_\infty^{\bar{\lambda}\to 0}(0)+\int_0^T\left[\iota_\infty^{\bar{\lambda}\to 0}(\tau)\,\psi_{ss}(\tau)-\delta f_\infty^{\bar{\lambda}\to 0}(\tau)\right]d\tau=0$, and $f_\infty^{\bar{\lambda}\to 0}(\tau)=\int_\tau^T\iota_\infty^{\bar{\lambda}\to 0}(s)\,ds$.

Proof. Consider the following limit:

$$\begin{aligned}\iota_{\infty}^{\bar{\lambda}\to 0}(\tau) &\equiv \lim_{\bar{\lambda}\to 0}\iota_{\infty}(\tau, r_{\infty}(\bar{\lambda}))\\ &= \lim_{\bar{\lambda}\to 0}\frac{\psi_{ss}(\tau)-v_{\infty}(\tau, r_{\infty}(\bar{\lambda}))}{\bar{\lambda}\psi_{ss}(\tau)}\\ &= \lim_{\bar{\lambda}\to 0}\frac{1}{\bar{\lambda}\psi(\tau)}\left[\frac{\delta\left(1-e^{-r_{ss}^{*}\tau}\right)}{r_{ss}^{*}}-\frac{\delta\left(1-e^{-r_{\infty}(\bar{\lambda})\tau}\right)}{r_{\infty}(\bar{\lambda})}+e^{-r_{ss}^{*}\tau}-e^{-r_{\infty}(\bar{\lambda})\tau}\right].\end{aligned}$$

This is a limit of the form $\frac{0}{0}$ as $\lim_{\bar{\lambda}\to 0} r_{\infty}(\bar{\lambda}) = r^{*}$.[54] We do not have an expression for $r_{\infty}(\bar{\lambda})$, so we cannot apply L'Hôpital's rule directly. Instead, we compute:

$$\lim_{\bar{\lambda}\to 0}\frac{\iota_{\infty}(\tau, r_{\infty}(\bar{\lambda}))}{\iota_{\infty}(T, r_{\infty}(\bar{\lambda}))} = \lim_{r_{\infty}(\bar{\lambda})\to r^{*}}\frac{\frac{\delta\left(1-e^{-r^{*}\tau}\right)}{r^{*}}-\frac{\delta\left(1-e^{-r_{\infty}(\bar{\lambda})\tau}\right)}{r}+e^{-r^{*}\tau}-e^{-r_{\infty}(\bar{\lambda})\tau}}{\frac{\delta\left(1-e^{-r^{*}T}\right)}{r^{*}}-\frac{\delta\left(1-e^{-r_{\infty}(\bar{\lambda})T}\right)}{r}+e^{-r^{*}T}-e^{-r_{\infty}(\bar{\lambda})T}}\frac{\psi_t}{\psi(\tau)},$$

which also has a limit of the form $\frac{0}{0}$. Now we can apply L'Hôpital's. We obtain:

$$\begin{aligned}\lim_{\bar{\lambda}\to 0}\frac{\iota_{\infty}(\tau, r_{\infty}(\bar{\lambda}))}{\iota_{\infty}(T, r_{\infty}(\bar{\lambda}))} &= \frac{\frac{-\delta r^{*}\tau e^{-r^{*}\tau}+\delta\left(1-e^{-r^{*}\tau}\right)}{r^{*2}}+\tau e^{-r^{*}\tau}}{\frac{-\delta r^{*}Te^{-r^{*}T}+\delta\left(1-e^{-r^{*}T}\right)}{r^{*2}}+Te^{-rT}}\frac{\psi_t}{\psi(\tau)}\\ &= \frac{1+\left[-1+\left(r^{*}/\delta-1\right)r^{*}\tau\right]e^{-r^{*}\tau}}{1+\left[-1+\left(r^{*}/\delta-1\right)r^{*}T\right]e^{-r^{*}T}}\frac{\psi_t}{\psi(\tau)}.\end{aligned}$$

If we define

$$\Xi \equiv \lim_{\bar{\lambda}\to 0}\iota_{\infty}(T, r_{\infty}(\bar{\lambda}))$$

then

$$\lim_{\bar{\lambda}\to 0}\iota_{\infty}(\tau, r_{\infty}(\bar{\lambda})) = \frac{1+\left[-1+\left(r^{*}/\delta-1\right)r^{*}\tau\right]e^{-r^{*}\tau}}{1+\left[-1+\left(r^{*}/\delta-1\right)r^{*}T\right]e^{-r^{*}T}}\frac{\psi_t}{\psi(\tau)}\Xi.$$

The value of κ then must be consistent with zero consumption:

$$y_{ss}-f_{\infty}^{\bar{\lambda}\to 0}(0)+\int_{0}^{T}\left[\iota_{\infty}^{\bar{\lambda}\to 0}(\tau)\,\psi_{ss}(\tau)-\delta f_{\infty}^{\bar{\lambda}\to 0}(\tau)\right]d\tau = 0,$$

for $f_{\infty}^{\bar{\lambda}\to 0}(\tau) = \int_{\tau}^{T}\iota_{\infty}^{\bar{\lambda}\to 0}(s)\,ds$. □

C.10 A further look into the revenue-effect

We analyze Ω to provide an interpretation of the revenue echo effect. We develop the explanation with the aid of figure 3. The revenue-echo Ω_t is the product of the marginal probability of default, $\theta(V_t)\,U'(\hat{c}_t)$, and an integral described below. To understand these terms, consider a small issuance of a (τ_0, t_0)-bond, located at the gray dot in figure 3. The bond matures as time progresses, as depicted in the figure by the gray ray in the direction $(1, -1)$. By time t, the bond has a maturity $\tau = \tau_0 - (t - t_0)$. Thus, the bond reflects the marginal impact on the repayment probability of a bond of maturity τ. If we multiply the term $\theta(V_t)\,U'(\hat{c}_t)$, inside Ω, by the term $\frac{U'(c_t)}{U'(\hat{c}_t)}v$ in the valuation, we obtain the marginal effect on the repayment probability of that bond:

[54] We drop the sub-index ss to ease the notation.

$$\underbrace{\theta\left(V_t\right)U'\left(c_t\right)}_{\text{effect on repayment probability}}\qquad \cdot v_t\left(\tau\right).$$

Hence, the term $\theta\left(V_t\right)U'\left(c_t\right)v_t\left(\tau\right)$ reflects the marginal effect of the (τ_0, t_0)-bond, on the repayment probability by time t.

The double integral captures how a lower repayment probability at time t impacts the revenues generated by all issuances in all moments prior to t. Notice how the range of integration covers all maturities $m \in [0, T]$ in the outer integral. The inner integral covers the relevant times prior to time t, $z \in \max\{t+m-T, 0\}$, when the marginal effect on repayment of the (t, τ)-bond affects the repayment probability of other bonds. These bonds are those that are still outstanding at time t, with a maturity m—represented graphically in the vertical line at time t. The effect of default at time t impacts past prices in the form of an "echo effect": Any bond price at a date prior to t for a bond that is still outstanding at time t should include the discounted value of its price at t, $e^{-\int_z^t(\hat{r}^*(u))du}\psi_t\left(m\right)$ for a specific maturity m. For example, the price of a bond with maturity m, $\psi_t\left(m\right)$, affects the price of all bonds of maturity $(m+t-z)$ at time z indexed by $z \in \max\{t+m-T, 0\}$. Each ray that extends from the vertical line at t depicts one such family of bonds. Thus, if we multiply the change in the repayment probability at t by $e^{-\int_z^t(\hat{r}^*(u))du}\psi_t\left(m\right)$, we obtain the reduction in the price of the $(m+t-z, z)$-bond. We can do the same for all bonds in $m \in [0, T]$, the outer integral, and past times z, the inner integral, to obtain the marginal effect that the (t, τ)-bond has on all past bond prices.

Naturally, the marginal impact on past prices affects past revenues. Thus, fix a maturity m and a date t. If we want to get the effect on revenues of a change in past prices, we must multiply the change in price by $\hat{\iota}\big(1-\left(\bar{\lambda}/2\right)\hat{\iota}\big)$, the issuance amount net of the liquidity costs. If we use the optimal issuance rule (15), revenues are proportional to:

$$\frac{1}{2\bar{\lambda}}\left[1-\left(\frac{\hat{v}_z\left(m+t-z\right)}{\hat{\psi}_z\left(m+t-z\right)}\right)^2\right].$$

Thus, when this term is inside the double integral, it captures the impact on past revenues of an increase in default probabilities. We bring past reductions in revenues into the current period t by multiplying by $e^{\int_z^t\hat{r}(u)du}$. The echo effect is present at any instant prior to the maturity of the bond, which is why it appears as a flow in equation (25). When the government considers the marginal issuance of the (τ_0, t_0)-bond, its valuation is the present value of all the echo effects Ω that last throughout the life of the bond. This reduction in revenues is part of the issuance consideration.

C.11 Proof of Proposition 6

Proof. Step 1. Setting the Lagrangian. Let $\mathbb{V}\left[\hat{f}_{t^o}\left(\cdot\right), X_{t^o}\right]$ denote the expected value of the government, at the instant t^o where the option to default is available, but prior to the decision of default. This value equals:

$$\mathbb{V}\left[\hat{f}_{t^o}\left(\cdot\right), X_{t^o}\right] = \mathbb{E}^X_{t^o}\left[\underbrace{\Gamma\left(V\left[\hat{f}_{t^o}\left(\cdot\right)\right], X_{t^o}\right)}_{\text{Default}} + \underbrace{\Theta\left(V\left[\hat{f}_{t^o}\left(\cdot\right), X_{t^o}\right]\right)V\left[\hat{f}_{t^o}\left(\cdot\right), X_{t^o}\right]}_{\text{No default}}\right],$$

where the first term in the expectation is the expected utility conditional on default given by $\Gamma(x) \equiv \int_x^\infty z d\Theta(z)$. The second term is the probability of no default time the perfect-foresight value. The Lagrangian is:

$$
\begin{aligned}
\mathcal{L}\left[\hat{\iota}, \hat{f}, \hat{\psi}\right] &= \mathbb{E}_{t^o}^X\left[\int_0^{t^o} e^{-\rho s} U\left(\hat{c}_s\right) ds + e^{-\rho t^o} \mathbb{V}\left[\hat{f}_{t^o}\left(\cdot\right), X_{t^o}\right]\right. \\
&+ \int_0^{t^o}\int_0^T e^{-\rho s} \hat{\jmath}_s\left(\tau\right)\left(-\frac{\partial \hat{f}}{\partial s} + \hat{\iota}_s\left(\tau\right) + \frac{\partial \hat{f}}{\partial \tau}\right) d\tau ds \\
&\left.+ \int_0^{t^o}\int_0^T e^{-\rho s} \hat{\mu}_s\left(\tau\right)\left(-\hat{r}^*\left(s\right)\hat{\psi}_s(\tau) + \delta + \frac{\partial \hat{\psi}}{\partial s} - \frac{\partial \hat{\psi}}{\partial \tau}\right) d\tau ds\right].
\end{aligned}
$$

In the Lagrangian, $\mathbb{E}^{t^o}$ denotes the conditional expectation with respect to the random time t^o. Here $\hat{\jmath}_s\left(\tau\right)$ and $\hat{\mu}_s\left(\tau\right)$ are the Lagrange multipliers. The first set of multipliers, $\hat{\jmath}_s\left(\tau\right)$, are associated with the law of motion of debt and appears also in previous sections. The second set of multipliers, $\hat{\mu}_s\left(\tau\right)$, are associated with the law of motion of bond prices. These terms appear because the government understands how its influence on the maturity profile affects the incentives to default, and hence impacts bond prices. This happens through the terminal condition:

$$
\begin{aligned}
\hat{\psi}_{t^o}(\tau) &= \mathbb{E}_{t^o}^X\left\{\Theta\left(V\left[\hat{f}_{t^o}\left(\cdot\right), X_{t^o}\right]\right)\psi_{t^o}(\tau)\right\}, \\
\hat{\psi}_t(0) &= 1.
\end{aligned}
$$

The terminal condition reflects that, at date t^o, the bond price is zero if default occurs. Otherwise it equals the perfect-foresight price, $\psi_{t^o}(\tau)$, if default does not occur.

Step 1.2. Re-writing the Lagrangian. Proceeding as in the proof of the deterministic case, as an intermediate step we integrate by parts the terms that involve time or maturity derivatives of $\hat{f}$ and $\hat{\psi}$. The Lagrangian $\mathcal{L}\left[\hat{\iota}, \hat{f}, \hat{\psi}\right]$ can thus be expressed as:

$$
\begin{aligned}
&\mathbb{E}_{t^o}^X\left[\int_0^{t^o} e^{-\rho s} U\left(\hat{c}_s\right) ds + e^{-\rho t^o} \mathbb{V}\left[\hat{f}_{t^o}\left(\cdot\right), X_{t^o}\right] - \int_0^T e^{-\rho t^o} \hat{f}_{t^o}\left(\tau\right)\hat{\jmath}_{t^o}(\tau) d\tau\right. \\
&+ \int_0^T \hat{f}_0\left(\tau\right)\hat{\jmath}_0\left(\tau\right) d\tau + \int_0^{t^o}\int_0^T e^{-\rho s}\hat{f}_s\left(\tau\right)\left(\frac{\partial \hat{\jmath}}{\partial s} - \rho \hat{\jmath}_s\left(\tau\right)\right) ds d\tau \\
&+ \int_0^{t^o} e^{-\rho s}\hat{f}_s\left(T\right)\hat{\jmath}_s\left(T\right) ds - \int_0^{t^o} e^{-\rho s}\hat{f}_0\left(s\right)\hat{\jmath}_0\left(0\right) ds \\
&- \int_0^{t^o}\int_0^T e^{-\rho s}\hat{f}_s\left(\tau\right)\frac{\partial \hat{\jmath}}{\partial \tau} d\tau ds + \int_0^{t^o}\int_0^T e^{-\rho s}\hat{\jmath}_s\left(\tau\right)\hat{\iota}_s\left(\tau\right) d\tau ds \\
&+ \int_0^{t^o}\int_0^T e^{-\rho s}\hat{\mu}_s\left(\tau\right)\left(-\hat{r}^*\left(s\right)\hat{\psi}_s(\tau) + \delta\right) d\tau ds, \\
&+ \int_0^T \left[e^{-\rho t^o}\hat{\mu}_{t^o}\left(\tau\right)\hat{\psi}_{t^o}(\tau) - \hat{\mu}_0\left(\tau\right)\hat{\psi}_0\left(\tau\right)\right] d\tau \\
&- \int_0^{t^o}\int_0^T e^{-\rho s}\hat{\psi}_s(\tau)\left(\frac{\partial \hat{\mu}}{\partial s} - \rho\hat{\mu}_s(\tau)\right) d\tau ds \\
&- \int_0^{t^o} e^{-\rho s}\left[\hat{\mu}_s\left(T\right)\hat{\psi}_s\left(T\right) - e^{-\rho s}\hat{\mu}_s\left(0\right)\hat{\psi}_s\left(0\right)\right] ds \\
&\left.+ \int_0^{t^o}\int_0^T e^{-\rho s}\hat{\psi}_s\left(\tau\right)\frac{\partial \hat{\mu}}{\partial \tau} d\tau ds\right].
\end{aligned}
$$

Step 1.3. Computing expectations. If we group terms, substitute the terminal conditions $f_s\left(T\right) = 0$ and $\hat{\psi}_{t^o}(\tau) = \Theta\left(V\left[\hat{f}_{t^o}\left(\cdot\right), X_{t^o}\right]\right)\psi_{t^o}(\tau)$

and compute the expected value with respect to t^o, we can express the Lagrangian $\mathcal{L}\left[\hat{\imath}, \hat{f}, \hat{\psi}\right]$ as:

$$
\begin{aligned}
&\int_0^\infty e^{-(\rho+\phi)s} U\left(\hat{c}_s\right) ds \\
&-\int_0^\infty e^{-(\rho+\phi)s} \hat{f}_0\left(s\right) \hat{\jmath}_s\left(0\right) ds \\
&-\int_0^\infty e^{-(\rho+\phi)s} \left[\hat{\mu}_s\left(T\right) \hat{\psi}_s\left(T\right) - \mu_s\left(0\right) \hat{\psi}_s\left(0\right)\right] ds \\
&+\int_0^\infty \int_0^T e^{-(\rho+\phi)s} \hat{f}_s\left(\tau\right) \left(\frac{\partial \hat{\jmath}}{\partial s} - \rho \hat{\jmath}_s\left(\tau\right)\right) ds d\tau \\
&-\int_0^\infty \int_0^T e^{-(\rho+\phi)s} \hat{f}_s\left(\tau\right) \frac{\partial \hat{\jmath}}{\partial \tau} d\tau ds \\
&+\int_0^\infty \int_0^T e^{-(\rho+\phi)s} \hat{\jmath}_s\left(\tau\right) \hat{\imath}_s\left(\tau\right) d\tau ds \\
&+\int_0^\infty \int_0^T e^{-(\rho+\phi)s} \hat{\mu}_s\left(\tau\right) \left(-\hat{r}_s^* \hat{\psi}_s(\tau) + \delta\right) d\tau ds \\
&-\int_0^\infty \int_0^T e^{-(\rho+\phi)s} \hat{\psi}_s(\tau) \left(\frac{\partial \hat{\mu}}{\partial s} - \rho \hat{\mu}_s(\tau)\right) d\tau ds \\
&+\int_0^\infty \int_0^T e^{-(\rho+\phi)s} \hat{\psi}_s\left(\tau\right) \frac{\partial \hat{\mu}}{\partial \tau} d\tau ds \\
&+\int_0^T f_0\left(\tau\right) \hat{\jmath}_0\left(\tau\right) d\tau - \int_0^T \hat{\mu}_0\left(\tau\right) \hat{\psi}_0\left(\tau\right) d\tau \\
&+\int_0^\infty e^{-(\rho+\phi)s} \phi \mathbb{V}\left[\hat{f}_s\left(\cdot\right), X_s\right] ds \\
&-\int_0^\infty \int_0^T e^{-(\rho+\phi)s} \phi \hat{f}_s\left(\tau\right) \hat{\jmath}_s\left(\tau\right) d\tau ds \\
&+\int_0^\infty \int_0^T e^{-(\rho+\phi)s} \phi \hat{\mu}_s\left(\tau\right) \mathbb{E}_s^X \left\{\theta\left(V\left[\hat{f}_s\left(\cdot\right), X_s\right]\right) \psi_s(\tau)\right\} d\tau ds.
\end{aligned}
$$

Next, we compute the Gâteaux derivatives with respect to each of the three arguments of the value function at a time.

Step 2. Computing the derivatives. Step 2.1. Gâteaux derivative with respect to the issuances. If we consider a perturbation around issuances and equalize it to zero, $\frac{\partial}{\partial \alpha} \mathcal{L}\left[\hat{\imath} + \alpha h, \hat{f}, \hat{\psi}\right]\Big|_{\alpha=0} = 0$, the result is identical to the risk-less case:

$$
U'\left(\hat{c}_t\right)\left(\frac{\partial q}{\partial \iota} \hat{\imath}_t\left(\tau\right) + q\left(t, \tau, \hat{\imath}\right)\right) = -\hat{\jmath}_t\left(\tau\right).
$$

Step 2.2. Gâteaux derivative with respect to the debt density. Since the distribution at the beginning $f_0\left(\tau\right)$ is given, any feasible perturbation must feature $h_0(\tau) = 0$ for any $\tau \in (0, T]$. In addition, we know that $h_t(T) = 0$, because $f_t(T) = 0$. The Gâteaux derivative of the continuation value with respect to the debt density is:

$$
\frac{d}{d\alpha} \mathbb{V}\left[\hat{f}_s\left(\cdot\right) + \alpha h_s\left(\cdot\right), X_s\right]\Big|_{\alpha=0} = \mathbb{E}_s^X \left\{\Theta\left(V\left[\hat{f}_s\left(\cdot\right), X_s\right]\right) \int_0^T j_s\left(\tau\right) h_s\left(\tau\right) d\tau\right\},
$$

where we have taken into account the fact that $\frac{d}{dx}\left(\Gamma\left(x\right) + \Theta\left(x\right) x\right) = \Theta\left(x\right)$ and $-$from the perfect foresight problem$-$:

$$
\frac{d}{d\alpha} V\left[\hat{f}_s\left(\cdot\right) + \alpha h_s\left(\cdot\right)\right]\Big|_{\alpha=0} = \int_0^T j_s\left(\tau\right) h_s\left(\tau\right) d\tau.
$$

Similarly, the Gâteaux derivative of the terminal bond price with respect to the debt density is

$$\frac{d}{d\alpha}\,\mathbb{E}_s^X\left\{\Theta\left(V\left[\hat{f}_s\left(\cdot\right)+\alpha h_s\left(\cdot\right),X_s\right]\right)\psi_s(\tau)\right\}\Big|_{\alpha=0}=\ldots$$
$$\mathbb{E}_s^X\left\{\theta\left(V\left[\hat{f}_s\left(\cdot\right),X_s\right]\right)\psi(\tau,s)\int_0^T j\left(\tau',s\right)h_s\left(\tau'\right)d\tau'\right\},$$

where $\theta(x)\equiv\frac{d}{dx}\Theta\left(x\right)$ is the probability density. The Gâteaux derivative of the Lagrangian with respect to the debt density, $\frac{d}{d\alpha}\,\mathcal{L}\left[\hat{\iota},\hat{f}+\alpha h,\hat{\psi}\right]\Big|_{\alpha=0}$, is thus:

$$\begin{aligned}
&\int_0^\infty e^{-(\rho+\phi)s}U'\left(\hat{c}_s\right)\left[-h_s\left(0\right)+\int_0^T\left(-\delta\right)h_s\left(\tau\right)d\tau\right]ds\\
&-\int_0^\infty e^{-(\rho+\phi)s}h_s\left(0,s\right)\hat{\jmath}_s\left(0\right)ds\\
&+\int_0^\infty\int_0^T e^{-(\rho+\phi)s}h_s\left(\tau\right)\left(\frac{\partial\hat{\jmath}}{\partial s}-\rho\hat{\jmath}_s\left(\tau\right)\right)dsd\tau\\
&-\int_0^\infty\int_0^T e^{-(\rho+\phi)s}h_s\left(\tau\right)\frac{\partial\hat{\jmath}}{\partial\tau}d\tau ds\\
&+\int_0^T h_0\left(\tau\right)\hat{\jmath}_0\left(\tau\right)d\tau\\
&+\int_0^\infty\int_0^T e^{-(\rho+\phi)s}\theta\mathbb{E}_s^X\left\{\Theta\left(V\left[\hat{f}_s\left(\cdot\right),X_s\right]\right)j_s\left(\tau\right)\right\}h_s\left(\tau\right)d\tau ds\\
&-\int_0^\infty\int_0^T e^{-(\rho+\phi)s}\phi h_s\left(\tau\right)\hat{\jmath}_s\left(\tau\right)d\tau ds\\
&+\int_0^\infty\int_0^T e^{-(\rho+\phi)s}\phi\mu_s\left(m\right)\mathbb{E}_s^X\left\{\theta\left(V\left[\hat{f}_s\left(\cdot\right),X_s\right]\right)\psi_s(m)\int_0^T j_s\left(\tau\right)h_s\left(\tau\right)d\tau\right\}dmds.
\end{aligned}$$

The value of the Gâteaux derivative of the Lagrangian for any perturbation, must be zero, i.e. $\frac{d}{d\alpha}\,\mathcal{L}\left[\hat{\iota},\hat{f}+\alpha h,\hat{\psi}\right]\Big|_{\alpha=0}=0$. Thus, a necessary condition is that all terms that multiply any entry of $h_s\left(\tau\right)$ add up to zero. We summarize the necessary conditions into:

$$\begin{aligned}
\rho\hat{\jmath}_s\left(\tau\right)&=\left(-\delta\right)U'\left(\hat{c}\left(s\right)\right)+\frac{\partial\hat{\jmath}}{\partial s}-\frac{\partial\hat{\jmath}}{\partial\tau} &(59)\\
&+\phi\mathbb{E}_s^X\left\{\left[\Theta\left(V\left[\hat{f}_s\left(\cdot\right),X_s\right]\right)+\theta\left(V\left[\hat{f}_s\left(\cdot\right),X_s\right]\right)\int_0^T\hat{\mu_s}\left(m\right)\psi_s(m)dm\right]j_s\left(\tau\right)-\hat{\jmath}_s\left(\tau\right)\right\}, &(60)\\
\hat{\jmath}_s\left(0\right)&=-U'\left(\hat{c}_s\right).
\end{aligned}$$

Step 2.3. Gâteaux derivative with respect to the bond price. In the case of the Gâteaux derivatives with respect to the evolution of the price $\hat{\psi}$, $\frac{d}{d\alpha}\,\mathcal{L}\left[\hat{\iota},f,\hat{\psi}+\alpha h\right]\Big|_{\alpha=0}$, we need to work first with the Lagragian before expectations have been computed. The reason is the following: only bonds that mature after default can be affected by the government's policies and hence the variations have to be zero for those bonds that mature before default, $\tilde{h}_t\left(\tau\right)=0$, if $\tau+t<t^o$. To incorporate this, we assume that admissible perturbations are of the form $\tilde{h}_t\left(\tau\right)=h_t\left(\tau\right)1_{\{\tau+t\geq t^o\}}$, where $h_t\left(\tau\right)$ is unrestricted. The Gâteaux derivative

is then

$$
\begin{aligned}
&\mathbb{E}^X_{t^o}\left[\int_0^{t^o} e^{-\rho s}U'\left(\hat{c}_s\right)\left(\int_0^T \hat{\iota}_s\left(\tau\right)\frac{\partial q}{\partial\hat{\psi}}1_{\{\tau+s\geq t^o\}}h_s\left(\tau\right)d\tau\right)ds\right.\\
&+\int_0^{t^o}\int_0^T e^{-\rho s}\hat{\mu}_s\left(\tau\right)\left(-\hat{r}^*_s 1_{\{\tau+s\geq t^o\}}h_s(\tau)\right)d\tau ds\\
&-\int_0^T \hat{\mu}_0\left(\tau\right)1_{\{\tau\geq t^o\}}h_0\left(\tau\right)d\tau\\
&-\int_0^{t^o}\int_0^T e^{-\rho s}1_{\{\tau+s\geq t^o\}}h_s(\tau)\left(\frac{\partial\hat{\mu}}{\partial s}-\rho\hat{\mu}_s(\tau)\right)d\tau ds\\
&-\int_0^{t^o} e^{-\rho s}\left[1_{\{T+s\geq t^o\}}h_s\left(T\right)\hat{\psi}_s\left(T\right)-e^{-\rho s}1_{\{s\geq t^o\}}h_s\left(0\right)\hat{\psi}_s\left(0\right)\right]ds\\
&\left.+\int_0^{t^o}\int_0^T e^{-\rho s}1_{\{\tau+s\geq t^o\}}h_s\left(\tau\right)\frac{\partial\hat{\mu}}{\partial\tau}d\tau ds\right].
\end{aligned}
$$

Note that the perturbation is only around $\hat{\psi}_s\left(\tau\right)$ and not $\psi_s\left(\tau\right)$, the terminal price after default, which is given. Since at maturity, bonds have a value of 1, $h_s\left(0\right)=0$, because no perturbation can affect that price. If we compute the expectation with respect to the random arrival time, t^o, we get:

$$
\begin{aligned}
&\int_0^\infty e^{-(\rho+\phi)s}\left(1-e^{-\phi\tau}\right)U'\left(\hat{c}_s\right)\left[\int_0^T \hat{\iota}_s\left(\tau\right)\frac{\partial q}{\partial\hat{\psi}}h_s\left(\tau\right)d\tau\right]ds\\
&+\int_0^\infty\int_0^T e^{-(\rho+\phi)s}\left(1-e^{-\phi\tau}\right)\hat{\mu}_s\left(\tau\right)\left(-\hat{r}^*_s h_s(\tau)\right)d\tau ds\\
&-\int_0^T\left(1-e^{-\phi\tau}\right)\mu_0\left(\tau\right)h_0\left(\tau\right)d\tau\\
&-\int_0^\infty\int_0^T e^{-(\rho+\phi)s}\left(1-e^{-\phi\tau}\right)h(\tau,s)\left(\frac{\partial\hat{\mu}}{\partial s}-\rho\mu_s(\tau)\right)d\tau ds\\
&-\int_0^\infty e^{-(\rho+\phi)s}\left(1-e^{-\phi T}\right)\mu_s\left(T\right)h_s\left(T\right)ds\\
&+\int_0^\infty\int_0^T e^{-(\rho+\phi)s}\left(1-e^{-\phi\tau}\right)\hat{\psi}_s\left(\tau\right)\frac{\partial\hat{\mu}}{\partial\tau}d\tau ds,
\end{aligned}
$$

where we use

$$
\mathbb{E}^X_{t^o}\left[1_{\{\tau+s>t^o>s\}}\right]=e^{-\phi s}\left(1-e^{-\phi\tau}\right).
$$

Again, as the Gâteaux derivative should be zero for any suitable $h(\tau,s)$, the optimality condition is

$$
\begin{aligned}
\left(\hat{r}^*_s-\rho\right)\hat{\mu}_s(\tau) &= U'\left(\hat{c}_s\right)\hat{\iota}_s\left(\tau\right)\frac{\partial q}{\partial\hat{\psi}}-\frac{\partial\hat{\mu}}{\partial s}+\frac{\partial\hat{\mu}}{\partial\tau},\\
\hat{\mu}_s\left(T\right) &= 0,\\
\hat{\mu}_0\left(\tau\right) &= 0.
\end{aligned}
$$

The solution to this PDE is

$$
\hat{\mu}_s\left(\tau\right)=\int_{\max\{s+\tau-T,0\}}^{s} e^{-\int_z^s(\hat{r}^*(u)-\rho)du}U'\left(\hat{c}_z\right)\hat{\iota}\left(\tau+s-z,z\right)\frac{\partial q_z}{\partial\hat{\psi}}\left(\tau+s-z\right)dz.
$$

If we integrate the discount factor of the government with respect to time, we obtain the following identity:

$$\int_z^s \hat{r}_u du = \int_z^s \rho du - \int_z^s \frac{U''(\hat{c}_u)}{U'(\hat{c}_u)} \hat{c}_u \frac{\dot{\hat{c}}_u}{\hat{c}_u} du.$$

Therefore, we have that

$$\int_z^s \hat{r}_u du - \int_z^s \rho du = -\log\left(U'(\hat{c}_u)\right)\Big|_z^s.$$

We obtain the following identity:

$$e^{\int_z^s \rho du} = e^{\int_z^s \hat{r}_u du} \frac{U'(\hat{c}_s)}{U'(\hat{c}_z)}.$$

Thus, the PDE for $\hat{\mu}_s(\tau)$ can be written as:

$$\hat{\mu}_s(\tau) = U'(\hat{c}_s) \int_{\max\{s+\tau-T,0\}}^s e^{-\int_z^s (\hat{r}_u^* - \hat{r}_u) du} \hat{\iota}_z(\tau+s-z) \frac{\partial q_z}{\partial \hat{\psi}}(\tau+s-z)\, dz.$$

Notice that

$$\begin{aligned}
\hat{\iota}\frac{\partial q}{\partial \hat{\psi}} &= \left(\hat{\iota} - \frac{\bar{\lambda}}{2}(\hat{\iota})^2\right) \\
&= \frac{1}{2\bar{\lambda}}\left(\frac{\hat{\psi}^2 - \hat{v}^2}{\hat{\psi}^2}\right) \\
&= \frac{1}{2\bar{\lambda}}\left(1 - \frac{\hat{v}^2}{\hat{\psi}^2}\right).
\end{aligned}$$

where the second line uses the optimal issuance rule. Hence, we can write the price multiplier as:

$$\hat{\mu}_s(\tau) = U'(\hat{c}_s) \int_{\max\{s+\tau-T,0\}}^s e^{-\int_z^s (\hat{r}_u^* - \hat{r}_u) du} \frac{1}{2\bar{\lambda}} \left(1 - \left(\frac{\hat{v}_z(\tau+s-z)}{\hat{\psi}_z(\tau+s-z)}\right)^2\right) dz.$$

We employ this solution in the main text.

Step 3: From Lagrange multipliers to valuations. We now employ the definitions of $\hat{v}_s(\tau) = -\hat{j}_s(\tau)/U'(\hat{c}_s)$ and $v_s(\tau) = -j_s(\tau)/U'(\hat{c}_s)$, we can express equations (59)-(60) as

$$\begin{aligned}
\hat{r}_s \hat{v}_s(\tau) &= \delta + \frac{\partial \hat{v}}{\partial s} - \frac{\partial \hat{v}}{\partial \tau} \\
&\quad + \phi \mathbb{E}_s^X \left\{ \left[\Theta(V_s) + \theta(V_s) \int_0^T \hat{\mu}_s(m) \psi_s(m) dm\right] \frac{U'(c_s)}{U'(\hat{c}_s)} v_s(\tau) - \hat{v}_s(\tau) \right\}, \\
\hat{v}_s(0) &= 1,
\end{aligned}$$

where we use the notation $\Theta(V_s) \equiv \Theta\left(V\left[\hat{f}_s(\cdot), X_s\right]\right)$ and $\theta(V(s)) \equiv \theta\left(V\left[\hat{f}_s(\cdot), X_s\right]\right)$. Therefore valuations can be expressed as

$$\begin{aligned}
\hat{v}_t(\tau) &= e^{-\int_t^{t+\tau} (\hat{r}_s + \phi) ds} \\
&\quad + \phi \int_t^{t+\tau} e^{-\int_t^s (\hat{r}_u + \phi) du} \left(\delta + \mathbb{E}_s^X \left[(\Theta(V_{t+s}) + \Omega_{t+s}) \frac{U'(c_{t+s})}{U'(\hat{c}_{t+s})} v_{t+s}(\tau - s)\right]\right) ds,
\end{aligned}$$

where

$$\Omega_t = \theta(V_t) \int_0^T \hat{\mu}_t(m) \psi_t(m) dm.$$

□

C.12 Proof of Proposition 7

Preliminary Calculations. Consider a RSS with $\delta = 0$ and $\bar{\lambda} \to \infty$. Note that in this case:

$$\lim_{\bar{\lambda}\to\infty} f_{ss}(\tau) = 0, \quad \lim_{\bar{\lambda}\to\infty} \iota_t(\tau) = 0,$$

The RSS valuations are given by:

$$\rho \hat{v}_{rss}(\tau) = -\frac{\partial \hat{v}_{rss}}{\partial \tau} + \phi \cdot \left(\mathbb{E}^X_{rss}\left[(\Theta_0 + \Omega_0) U'(y_0/y_{ss}) v_0(\tau)\right] - \hat{v}_{rss}(\tau)\right), \; \hat{v}_{rss}(0) = 1.$$

We can verify the following Lemma.

Lemma 2. *The solution to $\hat{v}_{rss}(\tau)$ is:*

$$\hat{v}_{rss}(\tau) = \exp(-(\rho+\phi)\tau) + \phi \int_0^\tau \exp(-(\rho+\phi)(\tau - s)) \mathbb{E}^X_{rss}\left[(\Theta_0 + \Omega_0) U'(y_0/y_{ss}) v_0(s)\right] ds.$$

Proof. We can verify the solution since:

$$\begin{aligned}
\frac{\partial \hat{v}_{rss}}{\partial \tau} &= -(\rho+\phi)\left(\underbrace{\exp(-(\rho+\phi)\tau) - (\rho+\phi)\phi \int_0^\tau \exp(-(\rho+\phi)(\tau-s)) \mathbb{E}^X_{rss}\left[(\Theta_{rss} + \Omega_{rss}) U'(y_0/y_{ss}) v_t(s)\right] ds}_{\hat{v}_{rss}(\tau)}\right) \\
&+ \phi \mathbb{E}^X_{rss}\left[(\Theta_0 + \Omega_0) U'(y_0/y_{ss}) v_0(\tau)\right].
\end{aligned}$$

Thus,

$$-\frac{\partial \hat{v}_{rss}}{\partial \tau} = (\rho+\phi)\hat{v}_{rss}(\tau) - \phi \mathbb{E}^X_{rss}\left[(\Theta_0 + \Omega_0) v_0(\tau) U'(y_0/y_{ss})\right].$$

□

Next, We consider a single event of a jump in income and investigate the special limit case $\alpha \to \infty$:

$$\lim_{\{\bar{\lambda},\alpha\}\to\infty} c_t = y_t = y_{ss} \; \forall t \neq t^o.$$

At t^o there is an instantaneous shock. Thus, y_t is characterized as a single impulse, and thus, a small jump in marginal utility $U'(y_0/y_{ss})$. Since, income is steady and equal to consumption for all periods other than t^o, we obtain:

$$v_0(\tau) = \exp(-\rho\tau).$$

We prove the following auxiliary Lemma.

Lemma 3. *The solution to $\hat{v}_{rss}(\tau)$ is:*

$$\lim_{\bar{\lambda}\to\infty} \lim_{\alpha\to\infty} \left[\hat{v}_{rss}(\tau)\right] = \exp(-(\rho+\phi)\tau)\left(1 + \mathbb{E}^X_{rss}\left[(\Theta_0 + \Omega_0) U'(y_0/y_{ss})\right](\exp(\phi\tau) - 1)\right).$$

Proof. Next observe that:

$$\begin{aligned}
\lim_{\bar{\lambda}\to\infty}\lim_{\alpha\to\infty}\mathbb{E}^{X}_{rss}\left[\left(\Theta_0+\Omega_0\right)U'\left(y_0/y_{ss}\right)\int_0^{\tau}\phi\exp\left(-\left(\rho+\phi\right)\left(\tau-s\right)\right)v_0\left(s\right)ds\right] &= \\
\exp\left(-\left(\rho+\phi\right)\tau\right)\mathbb{E}^{X}_{rss}\left[\left(\Theta_0+\Omega_0\right)\int_0^{\tau}\phi\exp\left(\left(\rho+\phi\right)\cdot s\right)U'\left(y_0/y_{ss}\right)\lim_{\bar{\lambda}\to\infty}\lim_{\alpha\to\infty}\left[v_t\left(s\right)\right]ds\right] &= \\
\exp\left(-\left(\rho+\phi\right)\tau\right)\mathbb{E}^{X}_{rss}\left[\left(\Theta_0+\Omega_0\right)U'\left(y_0/y_{ss}\right)\int_0^{\tau}\phi\exp\left(\left(\rho+\phi\right)\cdot s\right)\exp\left(-\rho s\right)ds\right] &= \\
\exp\left(-\left(\rho+\phi\right)\tau\right)\mathbb{E}^{X}_{rss}\left[\left(\Theta_0+\Omega_0\right)U'\left(y_0/y_{ss}\right)\int_0^{\tau}\phi\exp\left(\phi\cdot s\right)ds\right] &= \\
\mathbb{E}^{X}_{rss}\left[\left(\Theta_0+\Omega_0\right)U'\left(y_0/y_{ss}\right)\right]\exp\left(-\left(\rho+\phi\right)\tau\right)\left(\exp\left(\phi\tau\right)-1\right).
\end{aligned}$$

We make use of this expression below. □

Finally, we will obtain the result that all RSS issuance elasticities are proportional to:

$$Z\left(\tau\right)=\frac{1-\exp\left(-\phi\tau\right)}{\exp\left(\left(\rho-r^{*}_{ss}\right)\tau\right)-\left(1-\exp\left(-\phi\tau\right)\right)}>0.$$

Lemma 4. *The function $Z\left(\tau\right)$ is decreasing for any $\tau>T^{*}\equiv\frac{1}{\phi}\log\left(\frac{\phi+(\rho-r^{*}_{ss})}{(\rho-r^{*}_{ss})}\right)$.*

Proof. Thus function is of the form:

$$Z\left(\tau\right)=\frac{x\left(\tau\right)}{y\left(\tau\right)-x\left(\tau\right)},$$

and then

$$\begin{aligned}
\frac{\partial Z}{\partial\tau} &= Z\left(\tau\right)\left(\frac{1}{x}\frac{dx}{d\tau}-\frac{\frac{dy}{d\tau}}{y-x}+\frac{\frac{dx}{d\tau}}{y-x}\right)=Z\left(\tau\right)\left(\frac{dx}{d\tau}\left(\frac{y-x+x}{x\left(y-x\right)}\right)-\frac{dy}{d\tau}\frac{1}{y-x}\right)\\
&= \frac{Z\left(\tau\right)}{\left(y-x\right)}\cdot\left(\frac{dx}{d\tau}\left(\frac{y}{x}\right)-\frac{dy}{d\tau}\right),
\end{aligned}$$

where

$$x\left(\tau\right)\equiv1-\exp\left(-\phi\tau\right);\ y\left(\tau\right)\equiv\exp\left(\rho\tau\right),$$

and

$$y\left(\tau\right)-x\left(\tau\right)=\exp\left(\rho\tau\right)-1+\exp\left(-\phi\tau\right)>0.$$

Thus, we have that:

$$\begin{aligned}
sign\left[\frac{\partial Z}{\partial\tau}\right] &= sign\left[\frac{dx}{d\tau}\frac{1}{x}-\frac{dy}{d\tau}\frac{1}{y}\right].\\
&= sign\left[\frac{\phi\exp\left(-\phi\tau\right)}{1-\exp\left(-\phi\tau\right)}-\left(\rho-r^{*}_{ss}\right)\right]\\
&= sign\left[\phi\exp\left(-\phi\tau\right)-\left(\rho-r^{*}_{ss}\right)\left(1-\exp\left(-\phi\tau\right)\right)\right]
\end{aligned}$$

$\frac{\partial Z}{\partial\tau}\leq0$, if:

$$\frac{1}{\exp\left(\phi\tau\right)-1}<\frac{\left(\rho-r^{*}_{ss}\right)}{\phi}.$$

The function $\left(\exp\left(\phi\tau\right)-1\right)^{-1}$ is decreasing in τ and converges to infinity as $\tau\to0$. Thus, the function is decreasing for any

$\tau > \tau^*$ solving:

$$\frac{\phi + (\rho - r^*_{ss})}{(\rho - r^*_{ss})} < \exp(\phi T^*).$$

Which is solved by:

$$\tau^* = \frac{1}{\phi}\log\left(\frac{\phi + (\rho - r^*_{ss})}{(\rho - r^*_{ss})}\right) > 0.$$

□

We make use of these Lemma in the proof of the main result.

Part 1: Self-Insurance. We investigate the effect on the RSS issuances and WAM of a small drop in income from y_{ss} to y_0. We consider only a perturbation regarding income, that is:

$$y_0 = y_{ss}(1 - \varepsilon)$$

Also, by assumption $\Phi(\cdot) = 1$, and that $\Omega_{rss} = 0$.

$$\hat{v}_{rss}(\tau) = \exp(-(\rho + \phi)\tau) + U'((1 - \epsilon)) \int_0^\tau \phi \exp(-(\rho + \phi)(\tau - s)) v_0(s)\, ds.$$

Next, we evaluate the derivative of $v_{rss}(\tau)$ with respect to ε. We obtain:

$$\frac{\partial}{\partial \varepsilon}[\hat{v}_{rss}(\tau)]|_{\varepsilon=0} = \sigma \cdot \exp(-(\rho + \phi)\tau) \int_0^\tau \phi \exp((\rho + \phi)s) v_0(s)\, ds.$$

Then, since

$$\iota_{rss}(\tau) = \frac{1}{\bar{\lambda}}\left(1 - \frac{\hat{v}_{rss}(\tau)}{\hat{\psi}_{rss}(\tau)}\right)$$

we have that:

$$\frac{\partial}{\partial \varepsilon}[\iota_{rss}(\tau)]|_{\epsilon=0} = -\frac{\sigma}{\bar{\lambda}\psi_{rss}(\tau)} \cdot \exp(-(\rho + \phi)\tau) \int_0^\tau \phi \exp((\rho + \phi)\cdot s) v_0(s)\, ds < 0,$$

where $\hat{\psi}_{rss}(\tau) = \exp(-r^*_{ss}\tau)$. Thus, self-insurance reduces issuances. Next, we compute the semi-elasticity of issuances, with respect to the small deviation:

$$\epsilon^\tau_{rss,\epsilon} = \frac{1}{\iota_{rss}(\tau)}\frac{\partial}{\partial \varepsilon}[\iota_{rss}(\tau)]|_{\epsilon=0} = -\sigma \frac{\exp(-(\rho - r^*_{ss} + \phi)\tau) \int_0^\tau \phi \exp((\rho + \phi)\cdot s) v_0(s)\, ds}{1 - \left(\exp(-(\rho - r^*_{ss} + \phi)\tau) \int_0^\tau \phi \exp((\rho + \phi)\cdot s) v_0(s)\, ds\right)}.$$

Thus, the elasticity becomes:

$$\begin{aligned}
\lim_{\bar{\lambda}\to\infty}\lim_{\alpha\to\infty} \epsilon^\tau_{rss,\varepsilon} &= -\sigma \frac{\exp(-(\rho - r^*_{ss} + \phi)\tau)(\exp(\phi\tau) - 1)}{1 - \exp(-(\rho - r^*_{ss} + \phi)\tau)(\exp(\phi\tau) - 1)} \\
&= -\sigma \frac{\exp(\phi\tau) - 1}{\exp((\rho - r^*_{ss} + \phi)\tau) - (\exp(\phi\tau) - 1)} \\
&= -\sigma \frac{1 - \exp(-\phi\tau)}{\exp((\rho - r^*_{ss})\tau) + \exp(-\phi\tau) - 1} \\
&= -\sigma \cdot Z(\tau).
\end{aligned}$$

Thus, since $Z(\tau)$ is decreasing for $\tau > \tau^*$, the limit elasticity $\lim_{\bar{\lambda}\to\infty}\lim_{\alpha\to\infty} \epsilon^\tau_{rss,\epsilon}$ increasing. Thus, if the government can only issue at maturities $\tau > \tau^*$, self-insurance depresses issuances and increases the WAM.

Part 2: Credit-Risk. We now consider the limit where there are no shocks, and only the option to default is allowed. Let the Government be risk neutral. As $\bar{\lambda} \to \infty$, we get $\Omega_0 \approx 0$. Then,

$$\hat{v}_{rss}(\tau) = \exp(-(\rho+\phi)\tau) + \phi(\Theta_0+\varepsilon)\int_0^\tau \exp(-(\rho+\phi)(\tau-s))\, v_0(s)\, ds.$$

Likewise, the price is:

$$\hat{\psi}_{rss}(\tau) = \exp(-(r^*_{ss}+\phi)\tau) + \phi(\Theta_0+\varepsilon)\int_0^\tau \exp(-(r^*_{ss}+\phi)(\tau-s))\, \psi_0(s)\, ds.$$

Then, as show before, we have that:

$$\lim_{\bar{\lambda}\to\infty} v_t(s) = \exp(-\rho s)$$

and

$$\psi_t(s) = \exp(-r^*_{ss} s),$$

we obtain:

$$\hat{v}_{rss}(\tau) = \exp(-(\rho+\phi)\tau)(1+(\Theta_0+\varepsilon)(\exp(\phi\tau)-1)).$$

Thus:

$$\frac{\partial}{\partial\varepsilon}[\hat{v}_{rss}(\tau)]\,|_{\varepsilon=0} = \exp(-(\rho+\phi)\tau)(\exp(\phi\tau)-1)$$

and by analogy:

$$\frac{\partial}{\partial\varepsilon}\left[\hat{\psi}_{rss}(\tau)\right]|_{\varepsilon=0} = \exp(-(r^*_{ss}+\phi)\tau)(\exp(\phi\tau)-1)$$

Then:

$$\begin{aligned}
\frac{\partial}{\partial\varepsilon}[\iota_{rss}(\tau)]\,|_{\varepsilon=0} &= -\frac{1}{\bar{\lambda}}\left(\frac{\hat{v}_{rss}(\tau)}{\hat{\psi}_{rss}(\tau)}\right)\left(\frac{\frac{\partial}{\partial\varepsilon}[\hat{v}_{rss}(\tau)]}{\hat{v}_{rss}(\tau)} - \frac{\frac{\partial}{\partial\varepsilon}\left[\hat{\psi}_{rss}(\tau)\right]}{\hat{\psi}_{rss}(\tau)}\right)|_{\epsilon=0} \\
&= -\frac{1}{\bar{\lambda}}\left(\frac{\exp(-(\rho+\phi)\tau)}{\exp(-(r^*_{ss}+\phi)\tau)}\right)\left(\frac{\frac{\partial}{\partial\varepsilon}[\hat{v}_{rss}(\tau)]}{\hat{v}_{rss}(\tau)} - \frac{\frac{\partial}{\partial\varepsilon}\left[\hat{\psi}_{rss}(\tau)\right]}{\hat{\psi}_{rss}(\tau)}\right)|_{\epsilon=0} \\
&= 0,
\end{aligned}$$

where we have used that:

$$\frac{\frac{\partial}{\partial\varepsilon}[\hat{v}_{rss}(\tau)]}{\hat{v}_{rss}(\tau)}|_{\varepsilon=0} = \frac{\exp(-(\rho+\phi)\tau)(\exp(\phi\tau)-1)}{\exp(-(\rho+\phi)\tau)(1+(\Theta_0)(\exp(\phi\tau)-1))} = \frac{(\exp(\phi\tau)-1)}{1+\Theta_{rss}(\exp(\phi\tau)-1)}.$$

and

$$\frac{\frac{\partial}{\partial\varepsilon}\left[\hat{\psi}_{rss}(\tau)\right]}{\hat{\psi}_{rss}(\tau)}|_{\varepsilon=0} = \frac{(\exp(\phi\tau)-1)}{1+\Theta_{rss}(\exp(\phi\tau)-1)}.$$

Thus, issuances are unresponsive to changes in the default probability. Thus, the WAM does not change with credit risk.

Part 3: Revenue Echo. We now consider a shock to the revenue echo effect, while maintaining the assumption that $\sigma = 0$ and $\bar{\lambda} \to \infty$. As before we have that:

$$\begin{aligned}
\hat{v}_{rss}(\tau) &= \exp(-(\rho+\phi)\tau) + \phi(\Theta_0+\Omega_0+\varepsilon)\int_0^\tau \exp(-(\rho+\phi)(\tau-s))\, v_0(s)\, ds, \\
&= \exp(-(\rho+\phi)\tau)(1+(\Theta_0+\Omega_0+\varepsilon)(\exp(\phi\tau)-1))
\end{aligned}$$

with $\Omega_0 \approx 0$. Thus, the derivative is:

$$\lim_{\bar{\lambda}\to\infty} \frac{\partial}{\partial\varepsilon}\left[\hat{v}_{rss}(\tau)\right]|_{\varepsilon=0} = \exp\left(-(\rho+\phi)\tau\right)\left(\exp(\phi\tau)-1\right).$$

Bond prices are

$$\begin{aligned}\hat{\psi}_{rss}(\tau) &= \exp\left(-(r_{ss}^*+\phi)\tau\right)+\phi\Theta_0\int_0^\tau \exp\left(-(r_{ss}^*+\phi)(\tau-s)\right)\psi_0(s)\,ds \\ &= \exp\left(-(r_{ss}^*+\phi)\tau\right)\left(1+\Theta_0\left(\exp(\phi\tau)-1\right)\right)\end{aligned}$$

and

$$\frac{\partial}{\partial\varepsilon}\left[\hat{\psi}_{rss}(\tau)\right]|_{\varepsilon=0} = 0.$$

Issuances are

$$\begin{aligned}\iota_{rss}(\tau) &= \frac{1}{\bar{\lambda}}\left[1-\frac{\exp\left(-(\rho+\phi)\tau\right)\left(1+(\Theta_0+\Omega_0+\varepsilon)\left(\exp(\phi\tau)-1\right)\right)}{\exp\left(-(r_{ss}^*+\phi)\tau\right)\left(1+\Theta_0\left(\exp(\phi\tau)-1\right)\right)}\right] \\ &= \frac{1}{\bar{\lambda}}\left[1-\exp\left(-(\rho-r_{ss}^*)\tau\right)\right],\end{aligned}$$

where we have incorporated the fact that $\Omega_0 \approx 0$, $\varepsilon \approx 0$. Then, we have that:

$$\begin{aligned}\frac{\partial}{\partial\epsilon}\left[\iota_{rss}(\tau)\right]|_{\epsilon=0} &= -\frac{1}{\bar{\lambda}}\left(\frac{\frac{\partial}{\partial\varepsilon}\left[\hat{v}_{rss}(\tau)\right]}{\hat{\psi}_{rss}(\tau)}\right)|_{\epsilon=0}. \\ &= -\frac{1}{\bar{\lambda}}\frac{\exp\left(-(\rho+\phi)\tau\right)\left(\exp(\phi\tau)-1\right)}{\exp\left(-(r_{ss}^*+\phi)\tau\right)\left(1+\Theta_0\left(\exp(\phi\tau)-1\right)\right)} \\ &= -\frac{1}{\bar{\lambda}}\frac{\exp\left(-(\rho-r_{ss}^*)\tau\right)\left(\exp(\phi\tau)-1\right)}{\left(1+\Theta_0\left(\exp(\phi\tau)-1\right)\right)}\end{aligned}$$

Then, following previous steps, the semi-elasticity is

$$\lim_{\bar{\lambda}\to\infty} \epsilon_{rss,\varepsilon}^{\tau} = -\frac{\exp\left(-(\rho-r_{ss}^*)\tau\right)\left(\exp(\phi\tau)-1\right)}{\left[1-\exp\left(-(\rho-r_{ss}^*)\tau\right)\right]\left(1+\Theta_0\left(\exp(\phi\tau)-1\right)\right)} < 0$$

If $\Theta_0 \approx 1$, this simplifies to

$$\lim_{\bar{\lambda}\to\infty} \epsilon_{rss,\varepsilon}^{\tau} = -\frac{\left(1-\exp(-\phi\tau)\right)}{\left[\exp\left((\rho-r_{ss}^*)\tau\right)-1\right]}.$$

We compute the slope as .

$$\lim_{\bar{\lambda}\to\infty} \frac{\partial\epsilon_{rss,\varepsilon}^{\tau}}{\partial\tau} = \epsilon_{rss,\varepsilon}^{\tau}\left[\frac{\phi\exp(-\phi\tau)}{\left(1-\exp(-\phi\tau)\right)}-\frac{(\rho-r_{ss}^*)\exp\left((\rho-r_{ss}^*)\tau\right)}{\left[\exp\left((\rho-r_{ss}^*)\tau\right)-1\right]}\right].$$

This is positive if

$$\frac{\phi}{\left(\exp(\phi\tau)-1\right)} < \frac{(\rho-r_{ss}^*)}{\left[1-\exp\left(-(\rho-r_{ss}^*)\tau\right)\right]},$$

which holds for any τlarger than τ^{**} defined as

$$\frac{\phi}{\left(\exp(\phi\tau^{**})-1\right)} = \frac{(\rho-r_{ss}^*)}{\left[1-\exp\left(-(\rho-r_{ss}^*)\tau^{**}\right)\right]}.$$

It is trivial to check that $\tau^{**} = 0$. Hence the semi-elasticity is increasing and the WAM increases.

C.13 The case of default without liquidity costs: $\bar{\lambda} = 0$

We show here that the maturity structure is undetermined in the case without liquidity costs and a finite support of G. In proposition 14 below we show how, if distribution $\hat{f}^*$ is a solutions of Problem (24), then another distribution $\hat{f}'$ is also a solution provided that

$$\int_0^T \left(\psi_t(\tau, X_t) - \hat{\psi}_t(\tau)\right)\left(\hat{f}_t^*(\tau) - \hat{f}_t{}'(\tau)\right) d\tau = 0, \tag{61}$$

$$\int_0^T \left(\mathbb{E}_{X_t}\left[\Theta\left(V\left[\hat{f}_t^*(\cdot), X_t\right]\right)\psi_t(\tau, X_t)\right] - \hat{\psi}_t(\tau)\right)\left(\hat{f}_t^*(\tau) - \hat{f}_t{}'(\tau)\right) d\tau = 0. \tag{62}$$

Consider first the case of an income shock. Here X_t does not jump after the shock arrives. If the government decides to default then the maturity profile at the moment of default is irrelevant. If the government decides instead to repay, the post-shock yield curve will be $\psi_{t^o}(\tau)$, which differs from $\hat{\psi_{t^o}}(\tau)$ as the post-shock default premium is zero. The maturity structure is indeterminate because conditions (61) and (62) are two integral equation with a continuum of unknowns, $\hat{f_{t^o}}'(\tau)$.

Consider next the case of an interest rate shock, in which X_t jumps with the option to default. Condition (61) is a system of integral equations, indexed by X_{t^o}, where $\hat{f_{t^o}}'(\cdot)$ is the unknown. Provided that X_{t^o} may take N possible values, then we have at most N equations that need to be satisfied by the debt distribution. In addition we have equation (62) and the condition that the market debt should coincide. Notice that the number can be less than $N+2$ as in some states the government may default and then condition 61 is trivially satisfied for any debt profile that replicates the total debt at market prices before the shock arrival. In any case, the maturity structure is indeterminate.

The indeterminacy of the debt distribution in our model complements previous results in the literature. In particular, Aguiar et al. (Forthcoming) study a model of sovereign default similar to the one presented with the key difference that in their model the government cannot commit to future debt issuances whereas in our paper it ca, conditional on repayment. Aguiar et al. (Forthcoming) find how in that case the government only operates in the short end of the curve, making payments and retiring long-term bonds as they mature but never actively issuing or buying back such bonds. This is because short term bonds cannot be diluted. The authors also conjecture that the maturity structure would be indeterminate if the government had full commitment over its issuance path. This is precisely the case we study here, confirming their conjecture.

Proposition 14. *Let* $\{\hat{\iota}_t^*(\tau), \hat{f}_t^*(\tau), \hat{c}_t^*\}_{t\in[0,t^o]}$ *and* $\{\iota^*{}_t(\tau), f^*{}_t(\tau), c^*{}_t\}_{t\in(t^o,\infty)}$ *be the solution of Problem (24) when* $\lambda_t(\iota, \tau) = 0$. *Let* $\{\hat{\iota}'{}_t(\tau), \hat{f}_t{}'(\tau), \hat{c}'{}_t\}_{t\in[0,t^o]}$ *and* $\{\iota^*{}_t(\tau), f^*{}_t(\tau), c^*{}_t\}_{t\in(t^o,\infty)}$ *be such that, for every* $t \le t^o$ *and every value of* X_t,

$$\hat{B}_t^* = \hat{B}'{}_t \tag{63}$$

$$\int_0^T \left(\psi_t(\tau, X_t) - \hat{\psi}_t(\tau)\right)\left(\hat{f}_t^*(\tau) - \hat{f}_t{}'(\tau)\right) d\tau = 0, \tag{64}$$

$$\int_0^T \left(\mathbb{E}_{X_t}\left[\Theta\left(V\left[\hat{f}_t^*(\cdot), X_t\right]\right)\psi_t(\tau, X_t)\right] - \hat{\psi}_t(\tau)\right)\left(\hat{f}_t^*(\tau) - \hat{f}_t{}'(\tau)\right) d\tau = 0, \tag{65}$$

and $B_t^* = B_t'$ *for every* $t > t^o$. *Then,* $\hat{c}_t' = \hat{c}_t^*$ *and* $c_t' = c^*{}_t$. *Thus,* $\{\hat{\iota}'{}_t(\tau), \hat{f}_t{}'(\tau), \hat{c}'{}_t\}_{t\in[0,t^o]}$ *and* $\{\iota^*{}_t(\tau), f^*{}_t(\tau), c^*{}_t\}_{t\in(t^o,\infty)}$ *is also optimal.*

Proof. *Step 0. Default values*. The value functional of a policy given an initial debt $f_0(\cdot)$ is given by:

$$\hat{V}[f_0(\cdot)] = \mathbb{E}_0\left[\int_0^{t^o} e^{-\rho t}U(\hat{c}_t)\,dt + \mathbb{E}_{V^D,X_{t^o}}\left[e^{-\rho t^o}V^O\left[V_{t^o}^D, f_{t^o}(\cdot)\right]\right]\right]$$

where the post-default value $V^O\left[V_{t^o}^D, f_{t^o}(\cdot), X_{t^o}\right] \equiv \max\left\{V_{t^o}^D, V\left[f_{t^o}(\cdot), X_{t^o}\right]\right\}$ and $V\left[f_{t^o}(\cdot), X_{t^o}\right]$ is the value of the perfect-foresight solution. Note that, from the solution of the problem with perfect foresight, the value function only depends on the market value of total debt, $V\left[f_{t^o}(\cdot), X_{t^o}\right] = V\left(B_{t^o}(X_{t^o}), X_{t^o}\right)$, where $B_{t^o}(X_{t^o})$ is defined as the market value of debt

$$B_{t^o}(X_{t^o}) \equiv \int_0^T \psi_{t^o}(\tau, X_{t^o})\, f_{t^o}(\tau)d\tau.$$

Therefore, the post-default value

$$V^O\left[V_{t^o}^D, f_{t^o}(\cdot), X_{t^o}\right] = V^O\left(V_{t^o}^D, B_{t^o}(X_{t^o}), X_{t^o}\right),$$

also only depends on the aggregate market value of total debt, $B(t^o, X_{t^o})$. Because $B'_{t^o}(X_{t^o}) = B^*_{t^o}(X_{t^o})$ for every realization of X_{t^o} the default decision depends only on the market value of debt when the country receives the opportunity to default and not on the debt-maturity profile. Thus, continuation values are equal and it is enough to show that $\hat{c}_t{}^* = \hat{c}_t{}'$ for $t \le t^o$ to prove that the two policies yield the same utility.

Step 1. Pre-shock prices are equal. Pre-shock prices solve

$$\begin{aligned}
\hat{r}^*{}_t\hat{\psi}_t(\tau) &= \delta + \frac{\partial\hat{\psi}}{\partial t} - \frac{\partial\hat{\psi}}{\partial\tau} + \phi\mathbb{E}_t^X\left[\Theta\left(V\left[f_t(\cdot), X_t\right]\right)\psi_t(\tau, X_t) - \hat{\psi}_t(\tau)\right], \text{ if } t < t^o \\
\hat{\psi}_{t^o}(\tau) &= \mathbb{E}_{t^o}^X\left\{\Theta\left(V\left[\hat{f_{t^o}}(\cdot), X_{t^o}\right]\right)\psi_{t^o}(\tau)\right\} \\
\hat{\psi}_t(0) &= 1.
\end{aligned}$$

It holds that

$$\Theta\left(V\left[f_{t^o}^*(\cdot), X_{t^o}\right]\right) = \Theta\left(V\left(B_{t^o}^*(X_{t^o}), X_{t^o}\right)\right) = \Theta\left(V\left(B'_{t^o}(X_{t^o}), X_{t^o}\right)\right) = \Theta\left(V\left[f'_{t^o}(\cdot), X_{t^o}\right]\right).$$

This is a consequence of the fact that $V\left(B_{t^o}^*(X_{t^o}), X_{t^o}\right) = V\left(B'_{t^o}(X_{t^o}), X_{t^o}\right)$. Thus, pre-shock prices are equal for both policies.

Step 2. Law of of motion of debt before the shock arrival. By definition $\hat{B}_t = \int_0^T \hat{\psi}_t(\tau)\hat{f}_t(\tau)\,d\tau$. The dynamics of $\hat{B}_t$ for $t < t^o$ are:

$$d\hat{B}_t = \left(\int_0^T \left(\dot{\hat{\psi}}_t(\tau)\hat{f}_t(\tau) + \hat{\psi}_t(\tau)\dot{\hat{f}}_t(\tau)\right)d\tau\right)dt,$$

which, with similar derivations as in 11, yields to

$$d\hat{B}_t = \left(\hat{c}_t - y_t + \hat{r}^*{}_t\hat{B}_t + \phi\int_0^T\left(\mathbb{E}_{X_t}\left[\Theta\left(V\left(B_t(X_t), X_t\right)\right)\psi_t(\tau, X_t)\right] - \hat{\psi}_t(\tau)\right)\hat{f}_t(\tau)\,d\tau\right)dt.$$

Step 3. The expected jump. Note that (65) for all X_t implies that:

$$\phi\int_0^T\left(\mathbb{E}_{X_t}\left[\Theta\left(V\left(B_t^*(X_t), X_t\right)\right)\psi_t(\tau, X_t)\right] - \hat{\psi}_t(\tau)\right)\left(\hat{f}_t(\tau) - \hat{f}_t{}'(\tau)\right)d\tau = 0, \tag{66}$$

Combining this equation with the law of motion of debt we get that before the shock arrival, $t < t^o$,

$$d\hat{B}_t^{\ *} = d\hat{B}_t^{\ \prime}. \tag{67}$$

Step 4. The actual jump. Condition (64) guarantees that the jump is the same for any X_t if the country does not default. If it defaults, condition (64) is trivially satisfied as $\hat{B}_t^{\ *} = \hat{B}_t^{\ \prime}$ and the jump is also the same as market debt is then zero. Hence $\hat{B}^*_{t^o} = \hat{B}'_{t^o}$. Finally, taking all these results together we conclude that: $\hat{c}^*_t = \hat{c}'_t$ for all $t \le t^o$. As the policy $\{\hat{\iota}'_t(\tau), \hat{f}_t^{\ \prime}(\tau), \hat{c}'_t\}_{t\in[0,t^o]}$ and $\{\iota^*{}_t(\tau), f^*{}_t(\tau), c^*{}_t\}_{t\in(t^o,\infty)}$ achieves the same consumption path as the optimal, it is thus optimal. □

D Frictionless Cases: risk and default

Risk without liquidity costs We now consider the case without liquidity costs, $\bar{\lambda} = 0$ and no default, $\Theta = 1, \Omega = 0$. With positive liquidity costs, adjustments in portfolios are costly. By studying the problem at the limit where liquidity costs are zero, we can understand the extent to which the government can obtain insurance given the set of bonds it has available. Thus, it clarifies the extent to which liquidity costs limit insurance.

Toward that goal, we note that the necessary conditions of the problem are the same with and without liquidity costs, including the issuance rule. If the issuance rule holds, issuances are bounded if and only if valuations and prices are equal, $v = \psi$ and $\hat{v} = \hat{\psi}$. If we substitute $v = \psi$ and $\hat{v} = \hat{\psi}$ in the PDE for valuations, equation (25), and subtract the bond PDE from both sides, equation (22), we obtain a premium condition that must hold for all bonds:[55]

$$\hat{r}_t - \phi \mathbb{E}_t^X \left[\frac{U'(c_t)}{U'(\hat{c}_t)} \cdot \frac{\psi_t(\tau)}{\hat{\psi}_t(\tau)} \right] = \hat{r}^*{}_t - \phi \mathbb{E}_t^X \left[\frac{\psi_t(\tau)}{\hat{\psi}_t(\tau)} \right]. \tag{68}$$

The analysis of the different solutions of equation (68) provides useful information about the role of hedging and self-insurance. We analyze each case in turn.

Perfect hedging: replicating the complete-markets allocation. Equation (68) replicates the complete-markets allocation when consumption follows a continuous path, i.e., when $\hat{c}_{t^o} = c_{t^o}(X_{t^o})$ for any realization of the shock. This is because international investors are risk-neutral and the government is risk averse. If consumption does not jump, condition (68) implies $\hat{r} = \hat{r}^*$. In a complete markets economy consumption growth satisfies $\frac{\dot{c}_t}{c_t} = \frac{r_t^* - \rho}{\sigma}$, the same rule that it follows in a deterministic problem. Naturally, there is no RSS with positive consumption if $r_t^* < \rho$, but consumption converges asymptotically toward zero.

Consumption does not jump when it is possible to form a perfect hedge, a debt profile that generates a capital gain that exactly offsets the shock. Any shock changes the net-present value of income. Given the path of rates, the optimal consumption rule and the initial post-shock consumption produce a net-present value of consumption. A perfect hedge thus produces the capital gains such the net present value of consumption minus income at the time of the shock (denoted by $\Delta B_{t^o}(X_{t^o})$) is covered to the point where pre- and post-shock consumption are equal: $c_{t^o} = \hat{c_{t^o}}(X_{t^o})$. This must be true for any shock, X_{t^o}. In the context of the model, a perfect hedge exists if the debt distribution satisfies at all times t

$$\Delta B_t(X_t) = -\int_0^T \left(\psi_t(\tau; X_t) - \hat{\psi}_t(\tau) \right) \hat{f}_t(\tau)\, d\tau, \tag{69}$$

for any possible realization of X_t. This family of equations is a generalization of the discrete-shock and discrete-bonds matrix conditions that guarantee market completion in Duffie and Huang (1985), Angeletos (2002) or Buera and Nicolini (2004).[56]

[55] This equation is recovered also by solving the problem with $\lambda = 0$ directly. The proof is available upon request.

[56] In the case of discrete shocks and discrete bonds, the existence of complete-markets solution requires the presence of at least $N + 1$ bonds for N shocks. In the case of a continuum of shocks, the condition requires the invertibility of a linear operator.

In our model, perfect hedging is available in the case of an interest rate shock taking N possible values. Then, there is continuum of solutions that satisfy equation (69). In this case we can use a range of maturities $[0, T]$ that is as short as we want to hedge. The shorter the range, the more extreme the positions we obtain. A second observation has to do with the direction of hedges. Consider the case of a single jump in interest rates ($N = 1$). To offset the reduction in the net-present value of income, the debt profile must generate an increase in wealth. This requires an increase in short-term assets and long-term liabilities.

No hedging: only self insurance. The opposite to the complete-markets outcome is the case of income shocks. In this case bond prices do not change, $\psi = \hat{\psi} = 1$. Therefore, it is not possible to generate capital gains with a debt profile. Instead, the only solution to (68) is:

$$\frac{\dot{\hat{c}}_t}{\hat{c}_t} = \frac{\hat{r_t}^* + \phi\left(\mathbb{E}_t^X\left[\frac{U'(c_t)}{U'(\hat{c}_t)}\right] - 1\right) - \rho}{\sigma}.$$

This is a situation in which no hedging is available, because the asset space does not allow any form of external insurance. Instead, the government must self-insure. Self insurance is captured by the ratio of marginal utilities which effectively lowers $\hat{r}_t$. To solve for consumption, this extreme case coincides with a single-bond economy without interest-rate risk. The jump in consumption is given by the jump in the net present value of income. The solution to c_t in this case is known and can be found, for example in Wang, Wang and Yang (2016). The ratio of marginal utilities in the solution increases as the level of assets falls. This means that, provided there is a sufficiently low level of debt, the economy reaches a RSS with positive consumption. The convergence in consumption is a manifestation of self-insurance.

General case. The general case with both income and interest rates shocks described by equation (68) features an intermediate point between the two extreme cases described above as both a partial hedging and self-insurance emerge.[57] Furthermore, as long as the support of the shocks has cardinality N the debt profile is indeterminate, as only N points of the debt distribution are pinned down.[58]

D.1 Default without liquidity costs

We return to the case without liquidity costs, $\bar{\lambda} = 0$, but allow for default. Without liquidity costs, we have again that a solution necessarily features equality between valuations and prices, $v = \psi$ and $\hat{v} = \hat{\psi}$. As a result, the condition that characterizes the solution without default, (68), is modified to:

$$\hat{r}_t - \phi\mathbb{E}_t^X\left[\left(\Theta\left(V\left[\hat{f}_t(\cdot), X_t\right]\right) + \Omega_t\right)\frac{\psi_t(\tau)}{\hat{\psi}_t(\tau)} \cdot \frac{U'(c_t)}{U'(\hat{c}_t)}\right] = \hat{r}^*{}_t - \phi\mathbb{E}_t^X\left[\Theta\left(V\left[\hat{f}_t(\cdot), X_t\right]\right)\frac{\psi_t(\tau)}{\hat{\psi}_t(\tau)}\right]. \tag{70}$$

As in the case without default, we can explain how condition (70) characterizes the solution depending on the set of bonds and shocks.

On the impossibility of perfect hedging. The presence of default interrupts the ability to share risk. Efficient risk sharing requires a continuous consumption path along non-default states. To see how default interrupts risk-sharing, consider the case where interest-rate shocks allow complete asset spanning. Assume that $\hat{c}_t = c_t$ holds in non-default states, as in the

Proving conditions on G that guarantee that family of solutions exceeds the scope of the paper.In this case, equation (69) is just a system of N linear integral equations, for every t, known as Fredholm equations of the first kind.

[57] This case can be solved via dynamic programming using aggregate debt at market values $\hat{B}$ as a state variable. $\hat{B}$ is defined as in (49) with pre-shock prices and debt profile. Equation (68) holds for every maturity, so given $\hat{B}$, it represents a family of first-order conditions for f_t. The debt profile then is associated with an insurance cost of $\dot{\hat{B}}$.

[58] The proof is a particular case of the one presented in Appendix C.13 for the case with default.

version without default. In this case, condition (70) becomes

$$\hat{r_t} - \phi \mathbb{E}_t^X \left[\Omega_t \frac{\psi_t(\tau)}{\hat{\psi}_t(\tau)} \right] = \hat{r_t}^*.$$

This equation is not satisfied if two maturities feature a different price jump. However, full asset spanning requires a different price jump at two maturities. This contradiction implies that even when the set of securities can provide insurance in non-default states, the government's solution with commitment does not adopt a perfectly insuring scheme. The distortion follows because the echo-effect acts differently than the risk premium, it distorts valuations but not prices—we can see that even under risk-neutrality.

Default allows some hedging. Consider the case of only income shocks. Without default, we noted that there was no hedging role for maturity but now we show that with default, there is a role. Default opens the possibility of a partial hedging because prior to the shock, different maturities are priced differently. Post-shock prices are always $\psi_{t^o}(\tau) = 1$. This means that once a shock hits, the government can exploit the change in the yield curve to obtain capital gains in its portfolio. The change in the risk premium is akin to the spanning effect of an interest-rate jump.

General case. The option to default interrupts insurance across non-default states, but allows price variation even without interest-rate risk. As long as the cardinality of shocks is discrete, the maturity profile is indeterminate—a formal proof is found in Appendix C.13. One extreme case of indeterminacy is that of a shock which does not produce a jump in income nor interests, but only grants a default option. Aguiar et al. (Forthcoming) studies that shock in a discrete-time model similar to ours but without commitment.

E Computational method

We provide here a sketch of the numerical algorithm used to jointly solve for the equilibrium domestic valuation, $v_t(\tau)$, bond price, $q(t,\tau,\iota)$, consumption c_t, issuance $\iota_t(\tau)$ and density f_t in the perfect-foresight case. The initial distribution is $f_0(\tau)$. The algorithm proceeds in 3 steps. We describe each step in turn.

Step 1: Solution to the domestic value The steady state equation (12) is solved using an *upwind finite difference* scheme similar to Achdou, Han, Lasry, Lions and Moll (2017). We approximate the valuation $v_{ss}(\tau)$ on a finite grid with step $\Delta\tau$: $\tau \in \{\tau_1, ..., \tau_I\}$, where $\tau_i = \tau_{i-1} + \Delta\tau = \tau_1 + (i-1)\Delta\tau$ for $2 \leq i \leq I$. The bounds are $\tau_1 = \Delta\tau$ and $\tau_I = T$, such that $\Delta\tau = T/I$. We use the notation $v_i := v_{ss}(\tau_i)$, and similarly for the issuance $\iota_{i,}$. Notice first that the domestic valuation equation involves first derivatives of the valuations. At each point of the grid, the first derivative can be approximated with a forward or a backward approximation. In an upwind scheme, the choice of forward or backward derivative depends on the sign of the *drift function* for the state variable. As in our case, the drift is always negative, we employ a backward approximation in state:

$$\frac{\partial v(\tau_i)}{\partial \tau} \approx \frac{v_i - v_{i-1}}{\Delta\tau}. \tag{71}$$

The equation is approximated by the following upwind scheme,

$$\rho v_i = \delta + \frac{v_{i-1}}{\Delta\tau} - \frac{v_i}{\Delta\tau},$$

with terminal condition $v_0 = v(0) = 1$. This can be written in matrix notation as

$$\rho\mathbf{v} = \mathbf{u} + \mathbf{A}\mathbf{v},$$

where

$$\mathbf{A} = \frac{1}{\Delta\tau}\begin{bmatrix} -1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & -1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & -1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & \cdots & 1 & -1 & 0 \\ 0 & 0 & \cdots & 0 & 1 & -1 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_{I-1} \\ v_I \end{bmatrix}, \mathbf{u} = \begin{bmatrix} \delta - 1/\Delta\tau \\ \delta \\ \delta \\ \vdots \\ \delta \\ \delta \end{bmatrix}. \tag{72}$$

The solution is given by

$$\mathbf{v} = (\rho\mathbf{I} - \mathbf{A})^{-1}\mathbf{u}. \tag{73}$$

Most computer software packages, such as Matlab, include efficient routines to handle sparse matrices such as $\mathbf{A}$.

To analyze the transitional dynamics, define $t^{\max}$ as the time interval considered, which should be large enough to ensure a converge to the stationary distribution and time is discretized as $t_n = t_{n-1} + \Delta t$, in intervals of length

$$\Delta t = \frac{t^{\max}}{N-1},$$

where N is a constant. We use now the notation $v_i^n := v_{t_n}(\tau_i)$. The valuation at $t^{\max}$ is the stationary solution computed in (73) that we denote as $\mathbf{v}^N$. We choose a forward approximation in time. The dynamic value equation (12) can thus be expressed

$$r^n\mathbf{v}^n = \mathbf{u} + \mathbf{A}\mathbf{v}^n + \frac{\left(\mathbf{v}^{n+1} - \mathbf{v}^n\right)}{\Delta t},$$

where $r^n := r(t_n)$. By defining $\mathbf{B}^n = \left(\frac{1}{\Delta t} + r^n\right)\mathbf{I} - \mathbf{A}$ and $\mathbf{d}^{n+1} = \mathbf{u} + \frac{\mathbf{v}^{n+1}}{\Delta t}$, we have

$$\mathbf{v}^n = (\mathbf{B}^n)^{-1}\mathbf{d}^{n+1}, \tag{74}$$

which can be solved backwards from $n = N-1$ until $n = 1$.

The optimal issuance is given by

$$\iota_i^n = \frac{1}{\bar{\lambda}}\frac{(\psi_i^n - v_i^n)}{\psi_i^n},$$

where ψ_i^n is computed in an analogous form to v_i^n.

Step 2: Solution to the Kolmogorov Forward equation Analogously, the KFE equation (6) can be approximated as

$$\frac{f_i^n - f_i^{n-1}}{\Delta t} = \iota_i^n + \frac{f_{i+1}^n - f_i^n}{\Delta \tau},$$

where we have employed the notation $f_i^n := f_{t_n}(\tau_i)$. This can be written in matrix notation as:

$$\frac{\mathbf{f}^n - \mathbf{f}^{n-1}}{\Delta t} = \mathbf{i}^n + \mathbf{A}^{\mathbf{T}}\mathbf{f}^n, \tag{75}$$

where $\mathbf{A}^{\mathbf{T}}$ is the transpose of $\mathbf{A}$ and

$$\mathbf{f}^n = \begin{bmatrix} f_1^n \\ f_2^n \\ \vdots \\ f_{I-1}^n \\ f_I^n \end{bmatrix}, \mathbf{i}^n = \begin{bmatrix} \iota_1^n \\ \iota_2^n \\ \vdots \\ \iota_{I-1}^n \\ \iota_I^n \end{bmatrix}.$$

Given $\mathbf{f}_0$, the discretized approximation to the initial distribution $f_0(\tau)$, we can solve the KF equation forward as

$$\mathbf{f}_n = \left(\mathbf{I} - \Delta t \mathbf{A}^{\mathbf{T}}\right)^{-1}(\mathbf{i}^n \Delta t + \mathbf{f}_{n-1}), \ \ n = 1, .., N. \tag{76}$$

Step 3: Computation of Expenditure. The discretized budget constraint (7) can be expressed as

$$c^n = \bar{y}^n - f_1^{n-1} + \sum_{i=1}^{I}\left[\left(1_n - \frac{1}{2}\bar{\lambda}\iota_i^n\right)\iota_i^n\psi_i^n - \delta f_i^n\right]\Delta\tau, \ \ n = 1, .., N.$$

Compute

$$r^n = \rho + \frac{\sigma}{c^n}\frac{c^{n+1} - c^n}{\Delta t}, n = 1, .., N-1.$$

Complete algorithm The algorithm proceeds as follows. First guess an initial path for consumption, for example $c^n = \bar{y}^n$, for $n = 1, .., N$. Set $k = 1$;

Step 1: Issuances. Given c_{k-1} solve step 1 and obtain ι.

Step 2: KF. Given ι solve the KF equation with initial distribution f_0 and obtain the distribution f.

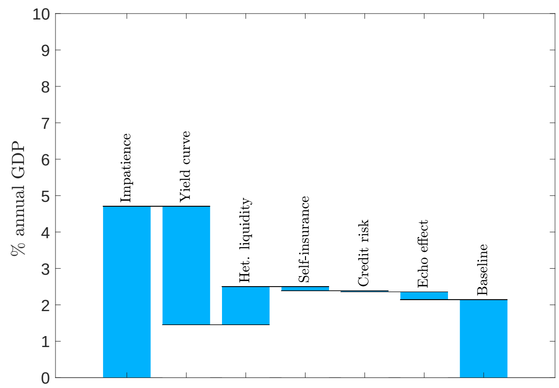
Step 3: Consumption. Given ι and f compute consumption c. If $\|c - c_{k-1}\| = \sum_{n=1}^{N}\left|c^n - c_{k-1}^n\right| < \varepsilon$ then stop. Otherwise compute

$$c_k = \omega c + (1-\omega)c_{k-1}, \ \lambda \in (0, 1),$$

set $k := k + 1$ and return to step 1.

The case with aggregate risk The method used to calculate a solution in the case with aggregate risk adds only one more unknown to the perfect-foresight algorithm; now we must obtain the RSS consumption. The idea is to propose a guess of c_{rss} and the path $\{c(t)\}$ and then compute the valuations using (25) for the RSS and (12) for the valuation after the shock. We then obtain issuances from the simple issuance rule (15). Given issuances, we compute the debt profile in the RSS using the law of motion of debt (6). The budget constraint (7) yields a new c_{rss} and a new path $\{c(t)\}$. Naturally, $\{c(t)\}$ is the solution to the perfect-foresight problem where $f(\tau, 0) = \hat{f}_{rss}(\tau)$.

F Additional figures



10-year bond

Figure 10: Force decomposition

Notes: A principal amounting to 10 percent of GDP expires at time 10.

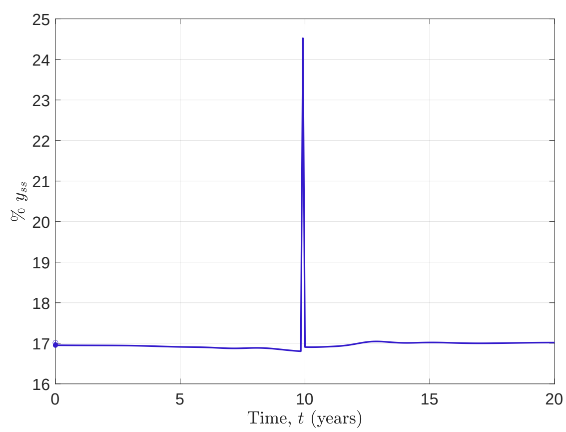


Figure 11: Issuances when a large principal is due to expire at year 10.

Notes: A principal amounting to 10 percent of GDP expires at time 10.