Appendix to Hébert and Woodford, "Rational Inattention with Sequential Information Sampling"

A Proofs

A.1 Proof of Lemma 1

The problem in the continuation region is (everywhere the value function is twice differentiable)

$$\sup_{\sigma_t \in M(q_t)} \frac{1}{2} tr[\sigma_t^T D(q_t) V_{qq}(q_t) D(q_t) \sigma_t] = \kappa,$$

subject to

$$\frac{1}{2} tr[\sigma_t^T k(q_t) \sigma_t] \leq \chi.$$

First, suppose that the constraint does not bind and a maximizing optimal policy exists:

$$\frac{1}{2} tr[\sigma_t^{*T} k(q_t) \sigma_t^*] = a\chi,$$

where σ_t^* is a maximizer, for some $a \in [0,1)$ ($a \geq 0$ by the positive semi-definiteness of $k(q_t)$). For any $c \in (1, a^{-1})$, with $a^{-1} = \infty$ for $a = 0$, if we used $\sigma_t = c\sigma_t^*$ instead, the policy would be feasible and we would have

$$\frac{1}{2} tr[\sigma_t^T D(q_t) V_{qq}(q_t) D(q_t) \sigma_t] = c^2 \kappa > \frac{1}{2} tr[\sigma_t^{*T} D(q_t) V_{qq}(q_t) D(q_t) \sigma_t^*] = \kappa,$$

a contradiction by the fact that $\kappa > 0$. Therefore, either the constraint binds under the optimal policy or an optimal policy does not exist. The latter would require that, for some vector $z \in \mathbb{R}^{|X|}$ with $zz^T \in M(q_t)$,

$$z^T D(q_t) V_{qq}(q_t) D(q_t) z > 0$$

and $z^T k(q_t) z = 0$, but the null space of $k(q_t)$ consists only of vectors whose elements are constant over the support of q_t, and therefore satisfy $q^T z \neq 0$, implying that $zz^T \notin M(q_t)$. Therefore, the constraint binds.

Using θ as defined in the lemma, it must be the case (anywhere the DM chooses not to stop and the value function is twice differentiable) that

$$\sup_{\sigma_t \in M(q_t)} \frac{1}{2} tr[\sigma_t \sigma_t^T (D(q_t) V_{qq}(q_t) D(q_t) - \theta k(q_t))] = 0.$$

Because of the homogeneity assumption on V,

$$q_t^T V_q(q_t) = V(q_t).$$

Differentiating again,

$$q_t^T V_{qq}(q_t) = 0.$$

It follows that, for any $\alpha \in \mathbb{R}$,

$$\frac{1}{2} tr[(\sigma_t \sigma_t^T + \alpha \iota \iota^T)(D(q_t) V_{qq}(q_t) D(q_t) - \theta k(q_t))] =$$
$$\frac{1}{2} tr[(\sigma_t \sigma_t^T)(D(q_t) V_{qq}(q_t) D(q_t) - \theta k(q_t))].$$

Suppose that we relax the requirement that $q_t^T \sigma_t = \vec{0}$ and simply require that σ_t by a square matrix. Let $\tilde{\sigma}_t$ be any square matrix. Setting

$$\alpha = -q_t^T \tilde{\sigma}_t \tilde{\sigma}_t^T q_t,$$

and performing an eigendecomposition,

$$VDV^T = \tilde{\sigma}_t \tilde{\sigma}_t^T + \alpha \iota \iota^T,$$

we construct a matrix

$$\sigma_t = VD^{\frac{1}{2}}$$

that achieves the same utility and satisfies $\sigma_t \in M(q_t)$. Therefore, ignoring this restriction is without loss of generality.

It immediately follows that, in the continuation region, the maximum eigenvalue of

$$D(q_t)V_{qq}(q_t)D(q_t) - \theta k(q_t)$$

must be equal to zero. If it were less than zero, we would always have

$$\frac{1}{2} tr[(\sigma_t \sigma_t^T)(D(q_t)V_{qq}(q_t)D(q_t) - \theta k(q_t))] < 0,$$

and if it were greater than zero, we could achieve

$$\frac{1}{2} tr[(\sigma_t \sigma_t^T)(D(q_t)V_{qq}(q_t)D(q_t) - \theta k(q_t))] > 0$$

by setting $\sigma_t = v_1 e_1^T$, where v_1 is an associated eigenvector of the maximal eigenvalue.

Finally, note that the DM would always choose to stop if $V(q_t) < \hat{u}(q_t)$, and therefore we must have $V(q_t) \geq \hat{u}(q_t)$. If $V(q_t) > \hat{u}(q_t)$, the DM must choose to continue, and (assuming twice-differentiability) the HJB equation must hold.

A.2 Proof of Theorem 1

Define $\phi(q_t)$ as the function described in the statement of the theorem (we will prove that it is indeed equal to $V(q_t)$, the value function of the dynamic problem). We will first show that $\phi(q_t)$ satisfies the HJB equation, can be implemented by a particular strategy for the DM, and that any other strategy for the DM achieves weakly less utility.

We begin by observing that

$$\iota^T k(q_t) D(q_t)^{-1} = 0 = \iota^T D(q_t) H_{qq}(q_t) = q_t^T H_{qq}(q_t).$$

We claim that, without loss of generality, we can assume that $H(q_t)$ is homogeneous of degree one,

$$H(\alpha q_t) = \alpha H(q_t)$$

for all $\alpha \in \mathbb{R}^+$ and $q_t \in \mathscr{P}(X)$. Differentiating with respect to α and then with respect to q_t, and evaluating at $\alpha = 1$, implies that

$$q_t^T H_{qq}(q_t) = 0,$$

consistent with the claim above.

We start by showing that the function $\phi(q_t)$ is twice-differentiable in certain directions. The function is

$$\phi(q_0) = \max_{\pi \in \mathscr{P}(A), \{q_a \in \mathscr{P}(X)\}_{a \in A}} \sum_{a \in A} \pi(a) u_a^T \cdot q_a - \theta \sum_{a \in A} \pi(a) D_H(q_a || q_0),$$

subject to the constraint that

$$\sum_{a\in A}\pi(a)q_a = q_0.$$

Substituting the definition of the divergence, we can rewrite the problem as

$$\phi(q_0) = \max_{\pi\in\mathscr{P}(A),\{q_a\in\mathscr{P}(X)\}_{a\in A}} \sum_{a\in A}\pi(a)u_a^T\cdot q_a + \theta H(q_0) - \theta\sum_{a\in A}\pi(a)H(q_a),$$

subject to the same constraint. Define a new choice variable,

$$\hat{q}_a = \pi(a)q_a.$$

By definition, $\hat{q}_a \in \mathbb{R}_+^{|X|}$, and the constraint is $\sum_{a\in A}\hat{q}_a = q_0$. By the homogeneity of H, the objective is

$$\phi(q_0) = \max_{\pi\in\mathscr{P}(A),\{q_a\in\mathscr{P}(X)\}_{a\in A},\{\hat{q}_a\in\mathscr{P}(X)\}_{a\in A}} \sum_{a\in A}u_a^T\cdot\hat{q}_a + \theta H(q_0) - \theta\sum_{a\in A}H(\hat{q}_a).$$

Any choice of $\hat{q}_a$ satisfying the constraint can be implemented by some choice of π and q_a in the following way: set

$$\pi(a) = \iota^T\hat{q}_a,$$

and (if $\pi(a) > 0$) set

$$q_a = \frac{\hat{q}_a}{\pi(a)}.$$

If $\pi(a) = 0$, set $q_a = q_0$. By construction, the constraint will require that $\pi(a) \leq 1$, $\sum_{a\in A}\pi(a) = 1$, and the fact that the elements of q_a are weakly positive will ensure $\pi(a) \geq 0$. Similarly, $\iota^T q_a = 1$ for all $a \in A$, and the elements of q_a are weakly greater than zero. Therefore, we can implement any set of $\hat{q}_a$ satisfying the constraints.

Rewriting the problem in Lagrangian form,

$$\phi(q_0) = \max_{\{\hat{q}_a \in \mathbb{R}^{|X|}\}_{a \in A}} \min_{\kappa \in \mathbb{R}^{|X|}, \{\nu_a \in \mathbb{R}_+^{|X|}\}_{a \in A}} \sum_{a \in A} u_a^T \cdot \hat{q}_a + \theta H(q_0)$$
$$- \theta \sum_{a \in A} H(\hat{q}_a) + \kappa^T (q_0 - \sum_{a \in A} \hat{q}_a) + \sum_{a \in A} \nu_a^T \hat{q}_a.$$

We begin by observing that $\phi(q_0)$ is convex in q_0. Suppose not: for some $q = \lambda q_0 + (1 - \lambda) q_1$, with $\lambda \in (0,1)$, $\phi(q) < \lambda \phi(q_0) + (1-\lambda)\phi(q_1)$. Consider a relaxed version of the problem in which the DM is allowed to choose two different $\hat{q}_a$ for each a. Observe that, because of the convexity of H, even with this option, the DM will set both of the $\hat{q}_a$ to the same value, and therefore the relaxed problem reaches the same value as the original problem. However, in the relaxed problem, choosing the optimal policies for q_0 and q_1 in the original problem, scaled by λ and $(1-\lambda)$ respectively, is feasible. It follows that $\phi(q) \geq \lambda \phi(q_0) + (1-\lambda)\phi(q_1)$. Note also that $\phi(q_0)$ is bounded on the interior of the simplex. It follows by Alexandrov's theorem that is is twice-differentiable almost everywhere on the interior of the simplex.

By the convexity of H, the objective function is concave, and the constraints are affine and a feasible point exists. Therefore, the KKT conditions are necessary. Moreover, the objective function is continuously differentiable in the choice variables and in q_0, and therefore the envelope theorem applies. We have, by the envelope theorem,

$$\phi_q(q_0) = \theta H_q(q_0) + \kappa,$$

and the first-order conditions (for all $a \in A$),

$$u_a - \theta H_q(\hat{q}_a) - \kappa + \nu_a = 0.$$

Define $\hat{q}_a(q_0)$, $\kappa(q_0)$, and $\nu_a(q_0)$ as functions that are solutions to the first-order conditions and constraints.

Consider an alternative prior, $\tilde{q}_0 \in \mathscr{P}(X)$, such that

$$\tilde{q}_0 = \sum_{a \in A} \alpha(a) \hat{q}_a(q_0)$$

for some $\alpha(a) \geq 0$. Conjecture that $\hat{q}_a(\tilde{q}_0) = \alpha(a)\hat{q}_a(q_0)$, $\kappa(\tilde{q}_0) = \kappa(q_0)$, and $\nu_a(\tilde{q}_0) = \nu_a(q_0)$. By the homogeneity property,

$$H_q(\alpha(a)\hat{q}_a(q_0)) = H_q(\hat{q}_a(q_0)),$$

and therefore the first-order conditions are satisfied. By construction, the constraint is satisfied, the complementary slackness conditions are satisfied, and $\hat{q}_a$ and ν_a are weakly positive. Therefore, all necessary conditions are satisfied, and by the concavity of the problem, this is sufficient. It follows that the conjecture is verified.

Consider a perturbation

$$q_0(\varepsilon; z) = q_0 + \varepsilon z,$$

with $z \in \mathbb{R}^{|X|}$, such that $q_0(\varepsilon; z)$ remains in $\mathscr{P}(X)$ for some $\varepsilon > 0$. If z is in the span of $\hat{q}_a(q_0)$, then there exists a sufficiently small $\varepsilon > 0$ such that the above conjecture applies. It follows in this case that κ is constant, and therefore $\phi_q(q_0(\varepsilon; z))$ is directionally differentiable with respect to ε. If $q_0(-\varepsilon; z) \in \mathscr{P}(X)$ for some $\varepsilon > 0$, then ϕ_q is differentiable, with

$$\phi_{qq}(q_0) \cdot z = \theta H_{qq}(q_0) \cdot z,$$

proving twice-differentiability in this direction. This perturbation exists anywhere the span of $\hat{q}_a(q_0)$ is strictly larger than the line segment connecting zero and q_0 (in other words, all

$\hat{q}_a(q_0)$ are not proportional to q_0). Define this region as the continuation region, Ω. Outside of this region, all $\hat{q}_a(q_0)$ are proportional to q_0, implying that

$$\phi(q_0) = \max_{a \in A} u_a^T \cdot q_0,$$

as required for the stopping region. Within the continuation region, the strict convexity of $H(q_0)$ in all directions orthogonal to q_0 implies that

$$\phi(q_0) > \max_{a \in A} u_a^T \cdot q_0,$$

as required.

Now consider an arbitrary perturbation z such that $q_0(\varepsilon; z) \in \mathbb{R}_+^{|X|}$ and $q_0(-\varepsilon; z) \in \mathbb{R}_+^{|X|}$ for some $\varepsilon > 0$. Observe that, by the constraint,

$$\varepsilon z = \sum_{a \in A} (\hat{q}_a(\varepsilon; z) - \hat{q}_a(q_0)).$$

It follows that

$$(\kappa^T(q_0(\varepsilon; z)) - \kappa^T(q_0))\varepsilon z = \sum_{a \in A} (\kappa^T(q_0(\varepsilon; z)) - \kappa^T(q_0))(\hat{q}_a(\varepsilon; z) - \hat{q}_a(q_0)).$$

By the first-order condition,

$$\begin{aligned} &(\kappa^T(q_0(\varepsilon; z)) - \kappa^T(q_0))(\hat{q}_a(\varepsilon; z) - \hat{q}_a(q_0)) = \\ &[\theta H_q(\hat{q}_a(q_0)) - \theta H_q(\hat{q}_a(\varepsilon; z)) + \nu_a^T(q_0(\varepsilon; z)) - \nu_a^T(q_0)](\hat{q}_a(\varepsilon; z) - \hat{q}_a(q_0)). \end{aligned}$$

Consider the term

$$(\nu_a^T(q_0(\varepsilon;z)) - \nu_a^T(q_0))(\hat{q}_a(\varepsilon;z) - \hat{q}_a(q_0)) = \sum_{x \in X} (\nu_a^T(q_0(\varepsilon;z)) - \nu_a^T(q_0)) e_x e_x^T (\hat{q}_a(\varepsilon;z) - \hat{q}_a(q_0)).$$

By the complementary slackness condition,

$$(\nu_a^T(q_0(\varepsilon;z)) - \nu_a^T(q_0))(\hat{q}_a(\varepsilon;z) - \hat{q}_a(q_0)) = -\nu_a^T(q_0(\varepsilon;z))\hat{q}_a(q_0) - \nu_a^T(q_0)\hat{q}_a(\varepsilon;z) \leq 0.$$

By the convexity of H,

$$\theta(H_q(\hat{q}_a(q_0)) - \theta H_q(\hat{q}_a(\varepsilon;z)))(\hat{q}_a(\varepsilon;z) - \hat{q}_a(q_0)) \leq 0.$$

Therefore,

$$(\kappa^T(q_0(\varepsilon;z)) - \kappa^T(q_0))\varepsilon z \leq 0.$$

It follows that anywhere ϕ is twice differentiable (almost everywhere on the interior of the simplex),

$$\phi_{qq}(q) \preceq \theta H_{qq}(q),$$

with equality in certain directions. Therefore, it satisfies the HJB equation almost everywhere in the continuation region. Moreover, by the convexity of ϕ,

$$(H_q(q_0(\varepsilon;z)) - H_q(q_0))^T \varepsilon z \geq (\phi_q(q_0(\varepsilon;z)) - \phi_q(q_0))^T \varepsilon z \geq 0,$$

implying that the "Hessian measure" (see Villani (2003)) associated with ϕ_{qq} has no pure point component. This implies that ϕ is continuously differentiable.

Next, we show that there is a strategy for the DM in the dynamic problem which can implement this value function. Suppose the DM starts with beliefs q_0, and generates some

$\hat{q}_a(q_0)$ as described above. As shown previously, this can be mapped into a policy $\pi(a, q_0)$ and $q_a(q_0)$, with the property that

$$\sum_{a \in A} \pi(a, q_0) q_a(q_0) = q_0.$$

We will construct a policy such that, for all times t,

$$q_t = \sum_{a \in A} \pi_t(a) q_a(q_0)$$

for some $\pi_t(a) \in \mathscr{P}(A)$. Let Ω (the continuation region) be the set of q_t such that a $\pi_t \in \mathscr{P}(A)$ satisfying the above property exists and $\pi_t(a) < 1$ for all $a \in A$. The associated stopping rule will be the stop whenever $\pi_t(a) = 1$ for some $a \in A$.

For all $q_t \in \Omega$, there is a linear map from $\mathscr{P}(A)$ to Ω, which we will denote $Q(q_0)$:

$$Q(q_0)\pi_t = q_t.$$

Therefore, we must have

$$Q(q_0) d\pi_t = D(q_t)\sigma_t dB_t.$$

By the assumption that $|X| \geq |A|$, there exists a $|A| \times |X|$ matrix $\sigma_{\pi,t}$ such that

$$Q(q_0)\sigma_{\pi,t} = D(q_t)\sigma_t$$

and

$$d\pi_t = \sigma_{\pi,t} dB_t.$$

Define

$$\tilde{\phi}(\pi_t) = \phi(q_t).$$

As shown above,

$$Q^T(q_0)\phi_{qq}(q_t)Q(q_0)$$

exists everywhere in Ω, and therefore

$$\tilde{\phi}(\pi_t) - \theta H(Q(q_0)\pi_t)$$

is a martingale.

We also have to scale $\sigma_{\pi,t}$ to respect the constraint,

$$\frac{1}{2}tr[\sigma_t \sigma_t^T k(q_t)] = \chi > 0.$$

This can be rewritten as

$$\frac{1}{2}tr[\sigma_{\pi,t}\sigma_{\pi,t}^T Q^T(q_0)D^+(Q(q_0)\pi_t)k(Q(q_0)\pi_t))D^+(Q(q_0)\pi_t)Q(q_0)] = \chi,$$

where D^+ denotes the pseudo-inverse.

By the positive-definiteness of k in all directions orthogonal to ι, we will always have $\frac{1}{2}tr[\sigma_{\pi,t}\sigma_{\pi,t}^T] > 0$. Under the stopping rule described previously, the boundary will be hit a.s. as the horizon goes to infinity. As a result, by the martingale property described above, initializing $\pi_0(a) = \pi(a, q_0)$,

$$\tilde{\phi}(\pi_0) = E_0[\tilde{\phi}(\pi_\tau) - \theta H(Q(q_0)\pi_\tau) + \theta H(Q(q_0)\pi_0)].$$

By Ito's lemma,

$$\theta H(Q(q_0)\pi_\tau) - \theta H(Q(q_0)\pi_0) = \int_0^\tau \chi\theta dt = \mu\tau.$$

By the value-matching property of ϕ, $\tilde{\phi}(\pi_\tau) = \hat{u}(Q(q_0)\pi_\tau)$. It follows that

$$\phi(q_0) = \tilde{\phi}(\pi_0) = E_0[\hat{u}(q_\tau) - \mu\tau],$$

as required.

Finally, we verify that alternative policies are sub-optimal. Consider an arbitrary control process σ_t and stopping rule described by the stopping time τ. We have, by the convexity of ϕ and the generalized Ito formula for convex functions (noting that we have shown that the Hessian measure associated with ϕ_{qq} has no pure point component), interpreting ϕ_{qq} in a distributional sense,

$$E_0[\phi(q_\tau)] - \phi(q_0) = \frac{1}{2}E_0[\int_0^\tau tr[\sigma_t^T D(q_t)\phi_{qq}(q_t)D(q_t)\sigma_t]dt].$$

By the feasibility of the policies, anywhere in the continuation region of the optimal policy,

$$\frac{1}{2}tr[\sigma_t^T D(q_t)\phi_{qq}(q_t)D(q_t)\sigma_t] \leq \frac{1}{2}\theta tr[\sigma_t^T k(q_t)\sigma_t] \leq \theta\chi.$$

In the stopping region of the optimal policy,

$$\frac{1}{2}tr[\sigma_t^T D(q_t)\phi_{qq}(q_t)D(q_t)\sigma_t] = 0 < \theta\chi.$$

Therefore,

$$\phi(q_0) \geq E_0[\phi(q_\tau)] - \int_0^\tau \theta\chi dt.$$

By the inequality

$$\phi(q_\tau) \geq \hat{u}(q_\tau),$$

we have

$$\phi(q_0) \geq E_0[\hat{u}(q_\tau) - \mu\tau]$$

for all policies, verifying optimality.

A.3 Proof of Lemma 2

Let p and p' be information structures with signal alphabet S. First, we will show that mixture feasibility and Blackwell monotonicity imply convexity. By mixture equivalence, letting p_M denote the mixture information structure and S_M the signal alphabet,

$$C(p_M, q; S_M) \leq \lambda C(p, q; S) + (1-\lambda)C(p', q; S).$$

Consider the garbling $\Pi : S \times \{1,2\} \to S$, which maps each $(s,i) \in S_M$ to $s \in S$. By Blackwell monotonicity,

$$C(p_M, q; S_M) \geq C(\Pi p_M, q; S).$$

By construction,

$$e_s^T \Pi p_M = \lambda e_s^T p + (1-\lambda) e_s^T p',$$

and the result follows.

Now we show the other direction: that convexity and Blackwell monotonicity imply mixture feasibility. Let p_1 and p_2 be information structures with signal alphabets S_1 and S_2. Because the cost function satisfies Blackwell monotonicity, it is invariant to Markov congruent embeddings. Define $S_M = (S_1 \cup S_2) \times \{1,2\}$. There exists an embedding Π_1 :

$S_1 \to S_M$ such that, for some $s_M = (s, i) \in S_M$,

$$e_{s_M}^T \Pi_1 p_1 = \begin{cases} 0 & i = 2 \\ 0 & s \notin S_1 \\ e_s^T p_1 & otherwise \end{cases} .$$

Define an embedding Π_2 along similar lines, and note that these embeddings are left-invertible. It follows by invariance that

$$C(\Pi_1 p_1, q; S_M) = C(p_1, q; S_1),$$

and likewise that

$$C(\Pi_2 p_2, q; S_M) = C(p_2, q; S_2).$$

By convexity,

$$C(\lambda \Pi_1 p_1 + (1-\lambda)\Pi_2 p_2; q; S_M) \leq \lambda C(\Pi_1 p_1, q; S_M) + (1-\lambda) C(\Pi_2 p_2, q; S_M).$$

Observing that

$$\lambda \Pi_1 p_1 + (1-\lambda)\Pi_2 p_2 = p_M$$

proves the result.

A.4 Proof of Theorem 2

Parts 1 and 2 of the theorem follow from a Taylor expansion of the cost function. Using the lemmas and theorem of Chentsov (1982), cited in the text, we know that for any invariant cost function with continuous second derivatives,

$$C(p,q;S)=\frac{1}{2}\Delta\sum_{x'\in X}\sum_{x\in X}(e_x^T k(q)e_{x'})\tau_{x'}^T g(r)\tau_x+o(\Delta).$$

The second claim follows by a similar argument.

We next demonstrate the claimed properties of $k(q)$. First, $k(q)$ is symmetric and continuous in q, by the symmetry of partial derivatives and the assumption of continuous second derivatives (Condition 4). Recall the assumption that

$$p_x=r+\Delta^{\frac{1}{2}}\tau_x+o(\Delta^{\frac{1}{2}}),$$

which implies that $\sum_{s\in S}e_s^T r=1$ and $\sum_{s\in S}e_s^T\tau_x=0$ for all $x\in X$. Consider an information structure for which $\tau_x=\phi e_x^T v$, where $v\in\mathbb{R}^{|X|}$ and $\phi\in\mathbb{R}^{|S|}$, with $\sum_{s\in S}e_s^T\phi=0$. Suppose that both v and ϕ are not zero. For this information structure,

$$C(p,q;S)=\frac{1}{2}\Delta\bar{g}v^T k(q)v+o(\Delta),$$

where $\phi^T g(r)\phi=\bar{g}>0$. Suppose the information structure is uninformative for all Δ. This would be the case if v is proportional to ι, and therefore

$$\iota^T k(q)\iota=0$$

by Condition 1. Regardless of whether the information structure is informative, by Condition 1, we must have

$$v^T k(q)v\geq 0,$$

implying that $k(q)$ is positive semi-definite. If z and $-z$ are in the tangent space of the simplex at q, there exists an x,x' $e_x^T z\neq e_{x'}^T z$ with x,x' in the support of q. Using z in the place of v above, by Condition 1, we must have

$$z^T k(q) z > 0.$$

Suppose now that the cost function satisfies Condition 5. Let v be as above, non-zero, and not proportional to ι. We have

$$C(p,q;S) = \frac{1}{2}\Delta \bar{g} v^T k(q) v + o(\Delta),$$

and therefore for the B defined in Condition 5 there exists a Δ_B such that, for all $\Delta < \Delta_B$, $C(p,q;S) < B$. Therefore, we must have

$$C(\{p_x\}_{x\in X}, q) \geq \frac{m}{2} \sum_{s\in S} (e_s^T pq) ||q_s - q||_X^2.$$

By Bayes' rule, for any signal that is received with positive probability,

$$q_s - q = \frac{(D(q) - qq^T) p^T e_s}{q^T p^T e_s}.$$

By convention, $q_s = q$ for any s such that $e_s^T pq = 0$.

The support of q_s is always a subset of the support of q, and therefore (by the equivalence of norms),

$$C(\{p_x\}_{x\in X}, q) \geq \frac{m_g}{2} \sum_{s\in S} (e_s^T pq)(q_s - q)^T D^+(q)(q_s - q)$$

for some constant $m_g > 0$.

For sufficiently large Δ, $e_s^T pq > 0$ if $e_s^T r_s > 0$, and therefore

$$C(\{p_x\}_{x\in X}, q) \geq \frac{m}{2} \sum_{s\in S: e_s^T r>0} \frac{(e_s^T p(D(q)-qq^T)D^+(q)(D(q)-qq^T)p^T e_s)}{(e_s^T pq)},$$

or,

$$C(\{p_x\}_{x\in X}, q) \geq \frac{m}{2}\Delta \sum_{s\in S: e_s^T r>0} (e_s^T \phi)^2 \frac{v^T(D(q)-qq^T)D^+(q)(D(q)-qq^T)v}{(e_s^T r)} + o(\Delta).$$

Noting that

$$\sum_{s\in S: e_s^T pq>0} \frac{(e_s^T \phi)^2}{(e_s^T pq)} = \phi^T g(r)\phi = \bar{g},$$

and that

$$(D(q)-qq^T)D^+(q)(D(q)-qq^T) = g^+(q),$$

we have

$$C(\{p_x\}_{x\in X}, q) \geq \frac{m_g}{2}\Delta \bar{g} v^T g^+(q) v + o(\Delta).$$

It follows that we must have

$$\frac{1}{2} v^T k(q) v \geq \frac{m_g}{2} v^T g^+(q) v$$

for all v.

A.5 Proof of Corollary 2

Under the stated assumptions,

$$p_x = r + \Delta^{\frac{1}{2}} \tau_x + o(\Delta^{\frac{1}{2}}).$$

By Bayes' rule, for any $s \in S$ such that $e_s^T pq > 0$,

$$q_s = \frac{D(q)p^T e_s}{q^T p^T e_s}.$$

It follows immediately that

$$\lim_{\Delta \to 0^+} q_s = D(q)\frac{r^T e_s}{r_s^T} = q.$$

Next,

$$\begin{aligned}\Delta^{-\frac{1}{2}}(q_s - q) &= \Delta^{-\frac{1}{2}} \frac{(D(q) - qq^T)p^T e_s}{q^T p^T e_s} \\ &= D(q)\frac{\tau^T e_s - \iota q^T \tau^T e_s + o(1)}{q^T p^T e_s}.\end{aligned}$$

For any s such that $q^T p^T e_s > 0$,

$$\lim_{\Delta \to 0^+} \Delta^{-\frac{1}{2}}(q_s - q) = D(q)\frac{\tau^T e_s - \iota q^T \tau^T e_s}{r^T e_s}.$$

By Theorem 2,

$$C(p,q;S) = \frac{1}{2}\Delta \sum_{x' \in X} \sum_{x \in X} (e_x^T k(q) e_{x'}) \tau_{x'}^T g(r) \tau_x + o(\Delta).$$

By the result that $\iota^T k(q) = 0$, we have

$$\begin{aligned}C(p,q;S) = \frac{1}{2}\Delta \sum_{x' \in X} \sum_{x \in X} e_x^T k(q) e_{x'} \cdot (\tau_{x'} - q\tau)^T g(r)(\tau_x - q\tau) \\ + o(\Delta).\end{aligned}$$

By the definition of the Fisher matrix, and the observation that $\iota^T \tau_x = 0$ for all $x \in X$,

$$(\tau_{x'} - q\tau)^T g(r)(\tau_x - q\tau) = \sum_{s \in S: e_s^T r > 0} (e_s^T r) \frac{(\tau_{x'} - q\tau)^T}{(e_s^T r)} e_s e_s^T \frac{(\tau_x - q\tau)}{(e_s^T r)}.$$

Substituting in the result regarding the posterior,

$$C(p,q;S) = \frac{1}{2} \sum_{s \in S: e_s^T r > 0} (e_s^T r)(q_s - q)^T D^+(q) k(q) D^+(q)(q_s - q) + o(\Delta),$$

which is the result.

A.6 Proof of Corollary 3

The cost function is directionally differentiable with respect to signals that add to the support of the signal distribution.

By directional differentiability and the continuity of the directional derivatives, there exists a function

$$f(\omega, r, q; S) = \lim_{\Delta \to 0^+} \frac{C(\bar{p}_\Delta + \Delta\omega, q; S) - C(\bar{p}_\Delta, q; S)}{\Delta}.$$

Observe that, if ωe_x is in the support of r for all x in the support of q, we must have $f(\omega, \bar{p}, q; S) = 0$, by the results of Theorem 2. Relatedly, if ω and ω' differ only with respect to the frequency of signals in the support of r for all x in the support of q, we must have

$$f(\omega, r, q; S) = f(\omega', r, q; S).$$

Assuming there are signals not in the support of $\bar{p}$, we can write $\omega = \omega_1 + \omega_2 + \ldots$, where each ω_i is a perturbation that contains only one signal not the support of $\bar{p}q$. Let

$N \leq |S|$ denote the number of these perturbations. We can define

$$f_i(\omega_i, r, q; S) = \lim_{\Delta \to 0^+} \frac{C(p_{i-1} + \Delta\omega_i, q; S) - C(p_{i-1}, q; S)}{\Delta},$$

where $p_{i-1} = \bar{p}_\Delta + \Delta \sum_{j=1}^{i-1} \omega_i$. By the assumption of the continuity of the directional derivatives,

$$f_i(\omega_i, r, q; S) = f(\omega_i, r, q; S).$$

It follows that

$$f(\omega, r, q; S) = \sum_{i=1}^{N} f(\omega_i, r, q; S).$$

By invariance, the function $f(\omega_i, r, q; S)$ does not depend on r or S. By the argument above, it is only a function of $e_{s_i}\omega_i$, where $s_i \in S$ is the unique signal in ω_i with $e_{s_i}^T r = 0$. By Bayes' rule,

$$e_{s_i}\omega_i = (e_{s_i}\omega_i q) D(q)^+ q_{s_i},$$

where q_{s_i} is the posterior associated with signal s_i. By the homogeneity of the directional derivative, we can rewrite this as

$$f(\omega_i, r, q; S) = (e_{s_i}\omega_i q) F(q_{s_i}, q).$$

By the requirement that the cost of an uninformative signal structure is zero, and everything else is costly, we must have

$$F(q, q) = 0,$$

$$F(q', q) > 0$$

for all $q' \neq q$. Therefore, F is a divergence, which we write $D^*(q'||q)$. The finiteness of $D^*(q'||q)$ is implied by the existence of the directional derivative. The approximation of the cost function follows from this result and Corollary 2.

By invariance, there exists a Markov congruent embedding that splits each signal in S into $M > 1$ distinct signals in S'. As M becomes arbitrarily large, the probability of each signal becomes small — and in particular, can be of order Δ. It follows for all $s \in S'$ such that $||q_s - q|| = O(\Delta^{\frac{1}{2}})$ (e.g. the signals described in Corollary 2), we must have

$$D^*(q_s||q) = \frac{1}{2}\Delta(q_s^T - q)\bar{k}(q)(q_s - q) + O(\Delta).$$

Moreover, by this argument, $D^*(q'||q)$ must be twice differentiable for q' in the neighborhood of q.

A.7 Proof of Lemma 3

We will show that Conditions 1-5 are satisfied. Recall the definition:

$$C_N(p,q;S) = \sum_{i \in \mathscr{I}(q)} \bar{q}_i \sum_{s \in S} e_s^T \bar{p}_i \, D_i(q_{i,s}||q_i)$$

A.7.1 Condition 1

Condition 1 requires that if the information structure is uninformative, the cost is zero, and if it is not, the cost is weakly positive. If the signal is uninformative, for any signal received with positive probability,

$$q_{i,s} = q_i,$$

and by our convention that $q_{i,s} = q_i$ if $\bar{q}_{i,s} = 0$, this also holds for zero-probability signals. By the definition of a divergence, $D_i(q_i||q_i) = 0$ for all q_i, and therefore the cost of an uninformative information structure is zero.

The cost is weakly positive by the definition of a divergence (being weakly positive) and the fact that probabilities are weakly positive.

A.7.2 Condition 2

Mixture feasibility requires that

$$C(p_M, q; S_M) \leq \lambda C(p_1, q; S_1) + (1-\lambda) C(p_2, q; S_2).$$

By definition,

$$\bar{p}_{i,M} = \frac{\sum_{x \in X_i} p_M e_x e_x^T q}{\bar{q}_i}$$

and

$$q_{i,s,M} = \frac{E_i q_{s,M}}{\sum_{x \in X_i} e_x^T q_{s,M}}$$

for any s such that $\bar{q}_{i,s,M} > 0$. For any $(s,1) \in S_M$, if $\bar{q}_{i,s,M} > 0$, we must have $\bar{q}_{i,s} > 0$, and therefore $q_{i,s,M} = q_{i,s,1}$ (denoting the posterior under p_1). The same argument holds for the second information structure.

It follows that

$$\begin{aligned}
C(p_M, q; S_M) &= \sum_{i \in \mathscr{I}(q)} \bar{q}_i \sum_{s \in S_M} e_s^T \bar{p}_{i,M} \, D_i(q_{i,s,M} || q_i) \\
&= \sum_{i \in \mathscr{I}(q)} \bar{q}_i \, (\lambda \sum_{s \in S_1} e_s^T \bar{p}_{i,1} \, D_i(q_{i,s,1} || q_i) + (1-\lambda) \sum_{s \in S_2} e_s^T \bar{p}_{i,2} \, D_i(q_{i,s,2} || q_i)) \\
&= \lambda C(p_1, q; S_1) + (1-\lambda) C(p_2, q; S_2),
\end{aligned}$$

verifying that the condition holds.

A.7.3 Condition 3

By Blackwell's theorem, for any Markov mapping $\Pi : S \to S'$, we require that

$$C(\Pi p, q; S') \leq C(p, q; S).$$

Consider a neighborhood $i \in \mathscr{I}(q)$. By definition,

$$\bar{p}_i' = \frac{\sum_{x \in X_i} \Pi p e_x e_x^T q}{\bar{q}_i} = \Pi \bar{p}_i$$

and

$$\begin{aligned} q_{i,s'} &= \frac{E_i q_{s'}}{\sum_{x \in X_i} e_x^T q_{s'}} \\ &= \frac{E_i D(q) p^T \Pi^T e_{s'}}{\sum_{x \in X_i} e_x^T D(q) p^T \Pi^T e_{s'}} \\ &= \frac{D(q_i) E_i p^T \Pi^T e_{s'}}{\bar{p}_i^T \Pi^T e_{s'}} \end{aligned}$$

where the second step follows by Bayes' rule,

$$D(q) p^T \Pi^T e_{s'} = (e_{s'}^T \Pi p q) q_{s'}.$$

Also by Bayes' rule,

$$\begin{aligned} D(q_i) E_i p^T e_s &= (e_s^T p E_i^T q_i) q_{i,s} \\ &= (e_s^T \bar{p}_i) q_{i,s}. \end{aligned}$$

and therefore

$$q_{i,s'} = \frac{\sum_{s\in S} q_{i,s}\bar{p}_i^T \Pi^T e_{s'}}{\bar{p}_i^T \Pi^T e_{s'}}.$$

It follows by the convexity of D_i in its first argument that

$$(\bar{p}_i^T \Pi^T e_{s'}) D_i(q_{i,s'}||q_i) \leq \sum_{s\in S} \bar{p}_i^T \Pi^T e_{s'} D_i(q_{i,s}||q_i).$$

Therefore,

$$\begin{aligned} C(\Pi p, q; S') &= \sum_{i\in\mathscr{I}(q)} \bar{q}_i \sum_{s'\in S'} e_{s'}^T \Pi \bar{p}_i \, D_i(q_{i,s'}||q_i) \\ &\leq \sum_{i\in\mathscr{I}(q)} \bar{q}_i \sum_{s'\in S'} \sum_{s\in S} \bar{p}_i^T \Pi^T e_{s'} D_i(q_{i,s}||q_i). \end{aligned}$$

By definition,

$$\sum_{s'\in S'} \Pi^T e_{s'} = 1$$

and therefore

$$C(\Pi p, q; S') \leq C(p, q; S).$$

A.7.4 Condition 4

By the definition of the neighborhood structure,

$$C_N(p, q; S) = \sum_{i\in\mathscr{I}(q)} \bar{q}_i \sum_{s\in S} e_s^T \bar{p}_i \, D_i(q_{i,s}||q_i),$$

and the twice-differentiability of D_i in its first argument, it is sufficient to show that $\bar{p}_i$ and $q_{i,s}$ are both twice-differentiable with respect to perturbations to p, in the neighborhood of an uninformative information structure.

Suppose that

$$p(\varepsilon) = r\iota^T + \varepsilon\tau + \nu\omega,$$

where $r \in \mathscr{P}(S)$ and the support of τe_x is in the support of r, and likewise for ωe_x, for all $x \in X$.

By Bayes' rule, for all $s \in S$ such that $e_s^T r > 0$,

$$q_s(\varepsilon,\nu) = \frac{D(q)p(\varepsilon,\nu)^T e_s}{q^T p(\varepsilon,\nu)^T e_s}.$$

Simplifying,

$$\begin{aligned} q_s(\varepsilon,\nu) = {} & q\frac{r^T e_s}{r^T e_s + \varepsilon q^T \tau^T e_s + \nu q^T \omega^T e_s} + \frac{\varepsilon D(q)\tau^T e_s}{r^T e_s + \varepsilon q^T \tau^T e_s + \nu q^T \omega^T e_s} \\ & + \frac{\nu D(q)\omega^T e_s}{r^T e_s + \varepsilon q^T \tau^T e_s + \nu q^T \omega^T e_s}. \end{aligned}$$

In the neighborhood around $\varepsilon = \nu = 0$, the denominator is strictly positive, and therefore

$$\frac{\partial}{\partial \nu} q_s(\varepsilon,\nu) = -q_s(\varepsilon,\nu)\frac{q^T \omega^T e_s}{r^T e_s + \varepsilon q^T \tau^T e_s + \nu q^T \omega^T e_s} + \frac{D(q)\omega^T e_s}{r^T e_s + \varepsilon q^T \tau^T e_s + \nu q^T \omega^T e_s}$$

and

$$\begin{aligned} \frac{\partial}{\partial \varepsilon}\frac{\partial}{\partial \nu} q_s(\varepsilon,\nu) = {} & q_s(\varepsilon,\nu)\frac{q^T \omega^T e_s}{r^T e_s + \varepsilon q^T \tau^T e_s + \nu q^T \omega^T e_s}\frac{q^T \tau^T e_s}{r^T e_s + \varepsilon q^T \tau^T e_s + \nu q^T \omega^T e_s} \\ & - \frac{q^T \omega^T e_s}{r^T e_s + \varepsilon q^T \tau^T e_s + \nu q^T \omega^T e_s}\frac{D(q)\tau^T e_s}{r^T e_s + \varepsilon q^T \tau^T e_s + \nu q^T \omega^T e_s} \\ & - q_s(\varepsilon,\nu)\frac{q^T \omega^T e_s}{r^T e_s + \varepsilon q^T \tau^T e_s + \nu q^T \omega^T e_s}\frac{q^T \tau^T e_s}{r^T e_s + \varepsilon q^T \tau^T e_s + \nu q^T \omega^T e_s} \\ & - \frac{D(q)\omega^T e_s}{r^T e_s + \varepsilon q^T \tau^T e_s + \nu q^T \omega^T e_s}\frac{q^T \tau^T e_s}{r^T e_s + \varepsilon q^T \tau^T e_s + \nu q^T \omega^T e_s}. \end{aligned}$$

For $s \in S$ such that $e_s^T r = 0$, $q_s(\varepsilon,\nu) = q$, and therefore $\frac{\partial}{\partial\varepsilon}\frac{\partial}{\partial\nu} q_s(\varepsilon,\nu) = 0$. Therefore,

$\frac{\partial}{\partial \nu} q_s(\varepsilon, \nu)$ can be written as a quadratic form in $vec(\tau)$ and $vec(\omega)$. It follows that $q_s(\varepsilon, \nu)$, in the neighborhood of an uninformative information structure, is twice-differentiable in the directions that do not change the support of the distribution of signals.

For all $i \in \mathscr{I}(q)$, define $\tilde{q}_i \in \mathscr{P}(X)$ as

$$e_x^T \tilde{q}_i = \begin{cases} \frac{e_x^T q}{\bar{q}_i} & x \in X_i \\ 0 & otherwise. \end{cases}$$

By definition,

$$\bar{p}_i(\varepsilon, \nu) = p\tilde{q}_i = r + \varepsilon \tau \tilde{q}_i + \nu \omega \tilde{q}_i.$$

and therefore is twice-differentiable in the required directions. Moreover, by construction, if $e_s^T r = 0$, then $e_s^T \bar{p}_i(\varepsilon, \nu) = 0$, and if $e_s^T r > 0$, then $e_s^T \bar{p}_i(\varepsilon, \nu) > 0$ in the neighborhood around $\varepsilon = \nu = 0$.

By definition,

$$q_{i,s}(\varepsilon, \nu) = \frac{E_i q_s(\varepsilon, \nu)}{\sum_{x \in X_i} e_x^T q_s(\varepsilon, \nu)}.$$

For all $i \in \mathscr{I}(q)$, in the neighborhood of an uninformative information structure, $\sum_{x \in X_i} e_x^T q_s(\varepsilon, \nu) \approx \bar{q}_i > 0$, and therefore the twice-differentiability of q_s in the required directions implies the twice-differentiability of $q_{i,s}$ in those directions.

A.7.5 Condition 5

This condition requires that, for some $m > 0$ and $B > 0$, for all $C(p, q; S) < B$,

$$C(p, q; S) \geq \frac{m}{2} \sum_{s \in S} (e_s^T pq) || q_s - q ||_X^2,$$

where $||\cdot||_X$ is an arbitrary norm on the tangent space of $\mathscr{P}(X)$. It follows immediately by the strong convexity of the divergence for the neighborhood that contains all states.

A.8 Proof of Lemma 4

Consider Corollary 2. Under the stated assumptions,

$$p_x = r + \Delta^{\frac{1}{2}} \tau_x + o(\Delta^{\frac{1}{2}})$$

$$q_{s,x} = q_x + \Delta^{\frac{1}{2}} q_x \frac{e_s^T (\tau_x - \sum_{x' \in X} \tau_{x'} q_{x'})}{e_s^T r} + o(\Delta^{\frac{1}{2}}).$$

By definition,

$$\bar{k}(q) = D^+(q) k(q) D^+(q),$$

and the cost function can be written as

$$C(\{p_x\}_{x\in X}, q; S) = \frac{1}{2} \sum_{s\in S} (e_s^T r)(q_s - q)^T \bar{k}(q)(q_s - q) + o(\Delta).$$

Now consider the definition of neighborhood cost function (20):

$$C_N(\{p_x\}_{x\in X}, q; S) \; = \sum_{i\in \mathscr{I}(q)} \bar{q}_i \sum_{s\in S} e_s^T \bar{p}_i \, D_i(q_{i,s}||q_i).$$

By definition,

$$\begin{aligned} \bar{q}_i \bar{p}_i &= \sum_{x\in X_i} p e_x e_x^T q \\ &= r\bar{q}_i + o(1). \end{aligned}$$

Note that

$$pq = r + o(1)$$

as well.

By Chentsov's theorem (Chentsov (1982)) and the approximation above,

$$D_i(q_{i,s}||q_i) = c_i(q_{i,s} - q_i)^T g(q_i)(q_{i,s} - q_i) + o(\Delta).$$

The approximation described in equation (21) follows.

Define the $|X| \times |X_i|$ matrix E_i that selects the elements of X that are contained in X_i. We have

$$\begin{aligned} q_{i,s,x} &= \frac{q_{s,x}(\Delta)}{\sum_{x' \in X_i} q_{s,x'}(\Delta)} \\ &= \frac{q_x}{\sum_{x' \in X_i} q_{x'}} + \Delta^{\frac{1}{2}} \frac{q_x}{\sum_{x' \in X_i} q_{x'}} \frac{e_s^T (\tau_x - \sum_{x' \in X} \tau_{x'} q_{x'})}{e_s^T r} \\ &\quad - \Delta^{\frac{1}{2}} \frac{q_x}{(\sum_{x' \in X_i} q_{x'})^2} \sum_{x' \in X_i} q_{x'} \frac{e_s^T (\tau_{x'} - \sum_{x'' \in X} \tau_{x''} q_{x''})}{e_s^T r} + o(\Delta^{\frac{1}{2}}). \end{aligned}$$

That is,

$$q_{i,s} = q_i + \frac{1}{\bar{q}_i} E_i(q_s - q) - \frac{1}{\bar{q}_i} q_i q_i^T D^+(q_i) E_i(q_s - q) + o(\Delta^{\frac{1}{2}}),$$

Using this,

$$\begin{aligned} &(q_{i,s} - q_i)^T g(q_i)(q_{i,s} - q_i) = (q_{i,s} - q_i)^T D^+(q_i)(q_{i,s} - q_i) \\ &= \frac{1}{(\bar{q}_i)^2}(q_s - q)^T E_i^T D^+(q_i) E_i(q_s - q^T) - \frac{1}{(\bar{q}_i)^2}(q_s - q)^T E_i^T D^+(q_i) q_i q_i^T D^+(q_i) E_i(q_s - q) \\ &\qquad - \frac{1}{(\bar{q}_i)^2}(q_s - q)^T E_i^T D^+(q_i) q_i q_i^T D^+(q_i) E_i(q_s - q) \\ &\qquad\qquad + \frac{1}{(\bar{q}_i)^2}(q_s - q)^T E_i^T D^+(q_i) q_i q_i^T D^+(q_i) E_i(q_s - q) + o(\Delta). \end{aligned}$$

Therefore,

$$\begin{aligned}
C_N(\{p_x\}_{x\in X}, q; S) &= \sum_{i\in\mathscr{I}(q)} c_i\bar{q}_i \sum_{s\in S}(e_s^T r)(q_{i,s}-q_i)^T g(q_i)(q_{i,s}-q_i) + o(\Delta) \\
&= \Delta \sum_{i\in\mathscr{I}(q)} c_i\bar{q}_i \sum_{s\in S}(e_s^T r)(q_s-q)^T \bar{k}_i(q)(q_s-q) + o(\Delta),
\end{aligned}$$

where

$$\bar{k}_i(q) = \frac{1}{(\bar{q}_i)^2} E_i^T (D^+(q_i) - D^+(q_i)q_i q_i^T D^+(q_i))E_i.$$

The $\bar{k}(q)$ matrix is

$$\begin{aligned}
\bar{k}_N(q) &= \sum_{i\in\mathscr{I}(q)} c_i\bar{q}_i\bar{k}_i(q) \\
&= \sum_{i\in\mathscr{I}(q)} \frac{c_i}{\bar{q}_i} E_i^T (D^+(q_i) - D^+(q_i)q_i q_i^T D^+(q_i))E_i. \qquad (30)
\end{aligned}$$

Thus, the associated $k(q)$ matrix is

$$\begin{aligned}
k_N(q) &= D(q)\bar{k}(q)D(q) \\
&= \sum_{i\in\mathscr{I}(q)} \frac{c_i}{\bar{q}_i} D(q)E_i^T (D^+(q_i) - D^+(q_i)q_i q_i^T D^+(q_i))E_i D(q) \\
&= \sum_{i\in\mathscr{I}(q)} \{c_i E_i^T D(q)E_i - c_i\bar{q}_i E_i^T q_i q_i^T E_i D\} \\
&= \sum_{i\in\mathscr{I}(q)} c_i\bar{q}_i E_i^T g^+(q_i)E_i.
\end{aligned}$$

A.9 Proof of Lemma 5

Using equation (30) from the proof of Lemma 4, we have

$$\bar{k}_N(q) = \sum_{i\in\mathscr{I}(q)} \frac{c_i}{\bar{q}_i} E_i^T (D^+(q_i) - D^+(q_i)q_i q_i^T D^+(q_i))E_i.$$

Consider the function

$$H_N(q) = \sum_{i \in \mathscr{I}(q)} c_i \left[\sum_{x \in X_i} (e_x^T q) \ln(e_x^T q) - (\sum_{x \in X_i} (e_x^T q)) \ln(\sum_{x \in X_i} (e_x^T q)) \right]$$
$$= \sum_{i \in \mathscr{I}(q)} c_i \sum_{x \in X_i} (e_x^T q) \ln(q_{i,x})$$
$$= -\sum_{i \in \mathscr{I}(q)} c_i \bar{q}_i H^{Shannon}(q_i).$$

Differentiating,

$$\frac{\partial H_N(q)}{\partial q_{x'}} = (\ln(q_{x'}) + 1) \sum_{i \in \mathscr{I}(q): x' \in X_i} c_i - \sum_{i \in \mathscr{I}(q): x' \in X_i} c_i (1 + \ln(\sum_{x \in X_i} (e_x^T q))).$$

Differentiating again,

$$\frac{\partial^2 H_N(q)}{\partial q_{x'} \partial q_{x''}} = \frac{\delta_{x',x''}}{q_{x'}} \sum_{i \in \mathscr{I}(q): x' \in X_i} c_i - \sum_{i \in \mathscr{I}(q): x', x'' \in X_i} \frac{c_i}{\sum_{x \in X_i} (e_x^T q)},$$

where $\delta_{x',x''}$ is the Kronecker delta. By definition,

$$\sum_{i \in \mathscr{I}(q)} \frac{c_i}{\bar{q}_i} e_{x'}^T E_i^T D^+(q_i) q_i q_i^T D^+(q_i) E_i e_{x''} = \sum_{i \in \mathscr{I}(q): x', x'' \in X_i} \frac{c_i}{\sum_{x \in X_i} (e_x^T q)}$$

and

$$\sum_{i \in \mathscr{I}(q)} \frac{c_i}{\bar{q}_i} e_{x'}^T E_i^T D^+(q_i) E_i e_{x''} = \delta_{x',x''} \sum_{i \in \mathscr{I}(q): x', x'' \in X_i} \frac{c_i}{(e_{x'}^T q)},$$

proving that $\bar{k}_N(q)$ is the Hessian of $H_N(q)$. Differentiation of $H_N(q)$ then yields the form given in the lemma for the associated Bregman divergence.

The posterior-separable static information-cost function is defined as

$$C_N^{static}(p, q; S) = \sum_{s \in S} (e_s^T p q)(H_N(q_s) - H_N(q)).$$

Using the definitions above,

$$C_N^{static}(p,q;S) = -\sum_{s\in S}(e_s^T pq)\sum_{i\in\mathscr{I}(q_s)} c_i\bar{q}_{i,s}H^{Shannon}(q_{i,s})$$
$$+\sum_{i\in\mathscr{I}(q)} c_i\bar{q}_i H^{Shannon}(q_i).$$

Note that $\bar{q}_{i,s} = 0$ for $i \in \mathscr{I}(q)\setminus\mathscr{I}(q_s)$, and $\mathscr{I}(q_s) \subseteq \mathscr{I}(q)$, and therefore

$$C_N^{static}(p,q;S) = -\sum_{s\in S}(e_s^T pq)\sum_{i\in\mathscr{I}(q)} c_i(\bar{q}_{i,s}H^{Shannon}(q_{i,s}) - \bar{q}_i H^{Shannon}(q_i)).$$

By Bayes' rule,

$$(e_s^T pq)\bar{q}_{i,s} = \bar{q}_i\bar{p}_{i,s}$$

and by definition,

$$\sum_{s\in S}\bar{p}_{i,s} = 1,$$

and therefore

$$C_N^{static}(p,q;S) = -\sum_{i\in\mathscr{I}(q)} c_i\bar{q}_i\sum_{s\in S}\bar{p}_{i,s}(H^{Shannon}(q_{i,s}) - H^{Shannon}(q_i))$$
$$= \sum_{i\in\mathscr{I}(q)} c_i\bar{q}_i\sum_{s\in S}\bar{p}_{i,s}D_{KL}(q_{i,s}||q_i).$$

The claim that

$$C_N^{static}(p,q;S) = \sum_{i\in\mathscr{I}(q)} c_i\sum_{x\in X:x\in X_i}(e_x^T q)D_{KL}(pe_x||pE_i^T q_i)$$

follows from the usual alternative ways of expressing mutual information and definitions.

A.10 Additional Definition and Lemmas

Definition 1. Let X^N be a sequence of state spaces, as described in section 5.2. A sequence of policies $\{p_N \in \mathscr{P}(X^N)\}_{N\in\mathbb{N}}$ satisfies the "convergence condition" if:

i) The sequence satisfies, for some constants $c_H > c_L > 0$, all N, and all $i \in X^N$,

$$\frac{c_H}{N+1} \geq e_i^T p_N \geq \frac{c_H}{N+1}.$$

ii) The sequence satisfies, for some constant $K_1 > 0$, all N, and all $i \in X^N \setminus \{0,N\}$,

$$N^3|\frac{1}{2}(e_{i+1}^T + e_{i-1}^T - 2e_i^T)p_N| \leq K_1,$$

and

$$N^2|\frac{1}{2}(e_N^T - e_{N-1}^T)p_N| \leq K_1$$

and

$$N^2|\frac{1}{2}(e_1^T - e_0^T)p_N| \leq K_1.$$

Lemma 11. *Given a function $p \in \mathscr{P}([0,1])$, define the sequence $\{p_N \in \mathscr{P}(X^N)\}_{N\in\mathbb{N}}$,*

$$e_i^T p_N = \int_{\frac{i}{N+1}}^{\frac{i+1}{N+1}} p(x)dx,$$

where X^N is the state space described in section 5.2. If the function p is strictly greater than zero for all $x \in [0,1]$, differentiable, and its derivative is Lipschitz continuous, then the sequence $\{p_N \in \mathscr{P}(X^N)\}_{N\in\mathbb{N}}$ satisfies the convergence condition, and satisfies, for some constant $K > 0$, all N, and all $i \in X^N \setminus \{0,N\}$,

$$N^2|\ln(\frac{1}{2}(e_{i+1}^T + e_i^T)q_N) + \ln(\frac{1}{2}(e_{i-1}^T + e_i^T)q_N) - 2\ln(e_i^T q_N)| \leq K$$

and

$$N|\ln(\frac{1}{2}(e_1^T+e_0^T)q_N)-\ln(e_0^Tq_N))|<K$$

and

$$N|\ln(\frac{1}{2}(e_N^T+e_{N-1}^T)q_N)-\ln(e_N^Tq_N))|<K.$$

Proof. The function p is strictly greater than zero, and continuous, and therefore attains a maximum and minimum on $[0,1]$, which we denote with c_H and c_L, respectively. By construction,

$$e_i^T p_N \geq \frac{c_L}{N+1}$$

and likewise for c_H, satisfying the bounds.

For all $i \in X^N \setminus \{N\}$,

$$\begin{aligned}(e_{i+1}^T - e_i^T)p_N &= \int_{\frac{i}{N+1}}^{\frac{i+1}{N+1}} (p(x+\frac{1}{N+1}) - p(x))dx \\ &= \int_{\frac{i}{N+1}}^{\frac{i+1}{N+1}} \int_0^{\frac{1}{N+1}} p'(x+y)dydx\end{aligned}$$

and therefore, letting K_2 be the maximum of the absolute value of p' on $[0,1]$ (which exists by the continuity of p'), we have

$$|(e_{i+1}^T - e_i^T)p_N| \leq \frac{1}{(N+1)^2}K_2,$$

satisfying the convergence condition for the endpoints.

For all $i \in X^N \setminus \{0,N\}$,

$$(e_{i+1}^T + e_{i-1}^T - 2e_i^T)p_N = \int_{\frac{i}{N+1}}^{\frac{i+1}{N+1}} (p(x+\frac{1}{N+1}) + p(x-\frac{1}{N+1}) - 2p(x))dx$$
$$= \int_{\frac{i}{N+1}}^{\frac{i+1}{N+1}} \int_0^{\frac{1}{N+1}} (p'(x+y) - p'(x-y))dydx.$$

By the Lipschitz continuity of p', it is absolutely continuous, and therefore

$$p'(x+y) = p'(x) + \int_0^y p''(x+z)dz.$$

It follows that

$$(e_{i+1}^T + e_{i-1}^T - 2e_i^T)p_N = \int_{\frac{i}{N+1}}^{\frac{i+1}{N+1}} \int_0^{\frac{1}{N+1}} \int_{-y}^{y} (p''(x+z))dzdydx.$$

Let K_3 denote the Lipschitz constant associated with p'. It follows that

$$|(e_{i+1}^T + e_{i-1}^T - 2e_i^T)p_N| \leq \frac{2K_3}{(N+1)^3}.$$

Therefore, the convergence condition is satisfied for $K = \max(\frac{1}{2}K_2, K_3)$.

By the concavity of the log function, and the inequality $\ln(x) \leq x-1$,

$$\ln(\frac{\frac{1}{2}(e_{i+1}^T + e_i^T)p_N}{e_i^T p_N}) + \ln(\frac{\frac{1}{2}(e_{i-1}^T + e_i^T)p_N}{e_i^T p_N}) \leq 2\ln(\frac{\frac{1}{4}(e_{i+1}^T + e_{i-1} + 2e_i^T)p_N}{e_i^T p_N})$$
$$\leq \frac{\frac{1}{2}(e_{i+1}^T + e_{i-1} - 2e_i^T)p_N}{e_i^T p_N}.$$

Therefore, by the bounds above,

$$\ln(\frac{\frac{1}{2}(e_{i+1}^T + e_i^T)p_N}{e_i^T p_N}) + \ln(\frac{\frac{1}{2}(e_{i-1}^T + e_i^T)p_N}{e_i^T p_N}) \leq \frac{(N+1)K}{N^3 c_L} \leq \frac{2K}{N^2 c_L}.$$

By the inequality $-\ln(\frac{1}{x}) \leq x-1$,

$$\ln(\frac{\frac{1}{2}(e_{i+1}^T + e_i^T)p_N}{e_i^T p_N}) + \ln(\frac{\frac{1}{2}(e_{i-1}^T + e_i^T)p_N}{e_i^T p_N}) \geq \frac{\frac{1}{2}(e_{i+1}^T - e_i^T)p_N}{\frac{1}{2}(e_{i+1}^T + e_i^T)p_N} + \frac{\frac{1}{2}(e_{i-1}^T - e_i^T)p_N}{\frac{1}{2}(e_{i-1}^T + e_i^T)p_N}.$$

We can rewrite this as

$$\begin{aligned}\ln(\frac{\frac{1}{2}(e_{i+1}^T + e_i^T)p_N}{e_i^T p_N}) + \ln(\frac{\frac{1}{2}(e_{i-1}^T + e_i^T)p_N}{e_i^T p_N}) \geq \\ (\frac{\frac{1}{2}(e_{i+1}^T + e_{i-1}^T - 2e_i^T)p_N}{\frac{1}{2}(e_{i+1}^T + e_i^T)p_N} + \frac{\frac{1}{2}(e_{i-1}^T - e_i^T)p_N}{\frac{1}{2}(e_{i+1}^T + e_i^T)p_N}(\frac{\frac{1}{2}(e_{i+1}^T + e_i^T)p_N}{\frac{1}{2}(e_{i-1}^T + e_i^T)p_N} - 1)).\end{aligned}$$

By the bounds above,

$$\frac{\frac{1}{2}(e_{i+1}^T + e_{i-1}^T - 2e_i^T)p_N}{\frac{1}{2}(e_{i+1}^T + e_i^T)p_N} \geq -\frac{2K}{N^2 c_L}$$

and

$$\begin{aligned}\frac{\frac{1}{2}(e_{i-1}^T - e_i^T)p_N}{\frac{1}{2}(e_{i+1}^T + e_i^T)p_N}(\frac{\frac{1}{2}(e_{i+1}^T + e_i^T)p_N}{\frac{1}{2}(e_{i-1}^T + e_i^T)p_N} - 1) &= \frac{\frac{1}{2}(e_{i-1}^T - e_i^T)p_N}{\frac{1}{2}(e_{i+1}^T + e_i^T)p_N}(\frac{\frac{1}{2}(e_{i+1}^T - e_{i-1}^T)p_N}{\frac{1}{2}(e_{i-1}^T + e_i^T)p_N}) \\ &\geq -\frac{N^2}{c_L^2}\frac{1}{(N+1)^4}(K_2)^2 \\ &\geq -(\frac{K_2}{2Nc_L})^2.\end{aligned}$$

Therefore,

$$N^2|\ln(\frac{\frac{1}{2}(e_{i+1}^T + e_i^T)p_N}{e_i^T p_N}) + \ln(\frac{\frac{1}{2}(e_{i-1}^T + e_i^T)p_N}{e_i^T p_N})| \leq \frac{2K}{c_L} + (\frac{K_2}{2c_L})^2.$$

For the end-points,

$$\frac{\frac{1}{2}(e_1^T - e_0^T)q_N}{\frac{1}{2}(e_1^T + e_0^T)q_N} \leq \ln(\frac{\frac{1}{2}(e_1^T + e_0^T)q_N}{e_0^T q_N}) \leq \frac{\frac{1}{2}(e_1^T - e_0^T)q_N}{e_0^T q_N}$$

and therefore

$$|\ln(\frac{\frac{1}{2}(e_1^T+e_0^T)q_N}{e_0^T q_N})| \leq \frac{K_2}{(N+1)c_L} \leq \frac{K_2}{Nc_L}.$$

A similar property holds for the other endpoint, and therefore the claim holds for $K_1 = \max(\frac{K_2}{c_L}, \frac{2K}{c_L} + (\frac{K_2}{2c_L})^2)$. □

Lemma 12. *Let $\{p_N \in \mathscr{P}(X^N)\}_{N\in\mathbb{N}}$ be a sequence of probability distributions over the state spaces associated with Theorem 3. Define the functions $\hat{p}_N \in \mathscr{P}([0,1])$ as, for $x \in [\frac{1}{2(N+1)}, 1-\frac{1}{2(N+1)})$,*

$$\hat{p}_N(x) = (N+1)((N+1)x+\frac{1}{2}-\lfloor(N+1)x+\frac{1}{2}\rfloor)e^T_{\lfloor(N+1)x+\frac{1}{2}\rfloor}p_N+ \\ +(N+1)(\frac{1}{2}-(N+1)x+\lfloor(N+1)x+\frac{1}{2}\rfloor)e^T_{\lfloor(N+1)x+\frac{1}{2}\rfloor-1}p_N,$$

and, for $x \in [0, \frac{1}{2(N+1)})$,

$$\hat{p}_N(x) = (N+1)e_0^T q_N,$$

and. for $x \in [1-\frac{1}{2(N+1)}, 1]$,

$$\hat{p}_N(x) = (N+1)e_N^T q_N.$$

If the sequence $\{p_N \in \mathscr{P}(X^N)\}_{N\in\mathbb{N}}$ satisfies the convergence condition (Definition 1), then there exists a sub-sequence, whose elements we denote by n, such that:

i) $p_n(x)$ converges point-wise to a differentiable function $p(x) \in \mathscr{P}([0,1])$, whose derivative is Lipschitz-continuous, with $p(x) > 0$ for all $x \in [0,1]$,

ii) the following sum converges:

$$\lim_{n\to\infty} n^2 \sum_{i\in X^n\setminus\{n\}} \{g(e_i^T p_N)+g(e_{i+1}^T p_N)-2g(\frac{1}{2}(e_i^T+e_{i+1}^T)p_N)\} = \frac{1}{4}\int_0^1 \frac{(p'(x))^2}{p(x)}dx,$$

where $g(x) = x\ln(x)$,

iii) for all $a \in A$, $\lim_{n\to\infty} u_{a,n}^T p_n = \int_0^1 u_a(x)p(x)dx$*, and*

iv) if the sequence $\{p_N \in \mathscr{P}(X^N)\}_{N\in\mathbb{N}}$ *is constructed from some function* $\tilde{p}(x)$*, as in Lemma 11, then* $p(x) = \tilde{p}(x)$ *for all* $x \in [0,1]$.

Proof. We begin by noting that the functions $\hat{p}_N(x)$ are absolutely continuous. Almost everywhere in $[\frac{1}{2(N+1)}, 1-\frac{1}{2(N+1)}]$,

$$\hat{p}_N'(x) = (N+1)^2(e^T_{\lfloor (N+1)x+\frac{1}{2}\rfloor} - e^T_{\lfloor (N+1)x+\frac{1}{2}\rfloor -1})p_N,$$

and outside this region, $\hat{p}_N'(x) = 0$. Let $f_N'(x)$ denote the right-continuous Lebesgue-integrable function on $[0,1]$ such that

$$\hat{p}_N(x) = \hat{p}_N(0) + \int_0^x f_N'(y)dy,$$

which is equal to $\hat{p}_N'(x)$ anywhere the latter exists.

The total variation of $f_N'(x)$ is equal to

$$TV(f_N') = \sum_{i=1}^{N-1}(N+1)^2|(e^T_{i+1} + e^T_{i-1} - 2e^T_i)p_N)| + \\ + (N+1)^2|(e^T_N - e^T_{N-1})p_N| + (N+1)^2|(e^T_1 - e^T_0)p_N|.$$

By the convergence condition,

$$TV(f_N') \leq \frac{(N+1)^3}{N^3}2K_1,$$

and therefore the sequence of functions $f_N'(x)$ has uniformly bounded variation. The function is also uniformly bounded at the end points, and therefore Helly's selection theorem applies. That is, there exists a sub-sequence, which we denote by n, such that $f_n'(x)$ converges point-wise to some $p'(x)$.

For any $1-\frac{1}{2(N+1)} > x > y \geq \frac{1}{2(N+1)}$, the quantity

$$\begin{aligned}|f_N'(x)-f_N'(y)| &= (N+1)^2 \left|\sum_{i=\lfloor (N+1)y+\frac{1}{2}\rfloor}^{\lfloor (N+1)x+\frac{1}{2}\rfloor} (e_{i+1}^T+e_{i-1}^T-2e_i^T)p_N\right| \\ &\leq \frac{(N+1)^2((N+1)(x-y)+2)}{N^3}2K_1.\end{aligned}$$

At the end points, for all $x \in [0, \frac{1}{2(N+1)})$,

$$|f_N'(\frac{1}{2(N+1)})-f_N'(x)| \leq \frac{2K_1}{N+1},$$

and for all $x \in [1-\frac{1}{2(N+1)}, 1]$,

$$|f_N'(x)-\lim_{y\uparrow 1-\frac{1}{2(N+1)}} f_N'(y)| \leq \frac{2K_1}{N+1}.$$

Therefore, by the point-wise convergence of f_n' to f_n', for all $x > y$,

$$|f'(x)-f'(y)| \leq 2K_1(x-y),$$

meaning that f' is Lipschitz-continuous. By the fact that $f'(0) = 0$, this implies that $|f'(x)| \leq 2K_1$ for all $x \in [0,1]$.

By the convergence condition, $c_L \leq \hat{p}_N(0) \leq c_H$. Therefore, there exists a convergent sub-sequence. We now use n to denote the sub-sequence for which $\lim_{n\to\infty} \hat{p}_n(0) = p(0)$ and for which $f_n'(x)$ converges point-wise to $p'(x)$. By the dominated convergence theorem, for all $x \in [0,1]$,

$$\lim_{n\to\infty} \hat{p}_n(x) = \lim_{n\to\infty}\{\hat{p}_n(0)+\int_0^x f_n'(y)dy\} = p(0)+\int_0^x p'(y)dy.$$

Define the function $p(x) = p(0) + \int_0^x p'(y)dy$ for all $x \in [0,1]$. By the convergence conditions, this function is bounded, $0 < c_L \leq p(x) \leq c_H$, by construction it is differentiable, and its derivative is Lipschitz continuous. Moreover,

$$\int_0^1 p(x)dx = 1,$$

and therefore $p \in \mathscr{P}([0,1])$.

Next, consider the limiting cost function. We have, Taylor-expanding,

$$g(y) = g(x) + g'(x)(y-x) + \frac{1}{2}g''(cy + (1-c)x)(y-x)^2$$

for some $c \in (0,1)$. Therefore,

$$\begin{aligned} g(e_i^T p_N) + g(e_{i+1}^T p_N) - 2g(\frac{1}{2}(e_i^T + e_{i+1}^T)p_N) = \\ \frac{1}{8}g''(c_1 e_i^T p_N + (1-c_1)\frac{1}{2}(e_i^T + e_{i+1}^T)p_N)((e_{i+1}^T - e_i^T)p_N)^2 \\ + \frac{1}{8}g''(c_2 e_i^T p_N + (1-c_2)\frac{1}{2}(e_i^T + e_{i+1}^T)p_N)((e_{i+1}^T - e_i^T)p_N)^2 \end{aligned}$$

for constants $c_1, c_2 \in (0,1)$. Note that, by the boundedness $\hat{p}_N(x)$ from below, $e_i^T p_N \geq (N+1)^{-1}c_L$ for all $i \in X^N$. It follows that

$$g''(c_1 e_i^T p_N + (1-c_1)\frac{1}{2}(e_i^T + e_{i+1}^T)p_N) = \frac{1}{c_1 e_i^T p_N + (1-c_1)\frac{1}{2}(e_i^T + e_{i+1}^T)p_N} \leq (N+1)c_L.$$

Therefore,

$$0 \leq g(e_i^T p_N) + g(e_{i+1}^T p_N) - 2g(\frac{1}{2}(e_i^T + e_{i+1}^T)p_N) \leq \frac{(N+1)c_L}{4}((e_{i+1}^T - e_i^T)p_N)^2.$$

By construction,

$$e_i^T p_N = \frac{1}{(N+1)} \hat{p}_N(\frac{2i+1}{2(N+1)}).$$

Therefore,

$$(N+1)(g(e_i^T p_N) + g(e_{i+1}^T p_N) - 2g(\frac{1}{2}(e_i^T + e_{i+1}^T) p_N)) =$$
$$g(\hat{p}_N(\frac{2i+1}{2(N+1)})) + g(\hat{p}_N(\frac{2i+3}{2(N+1)})) - 2g(\hat{p}_N(\frac{2i+2}{2(N+1)})).$$

and

$$g(e_i^T p_N) + g(e_{i+1}^T p_N) - 2g(\frac{1}{2}(e_i^T + e_{i+1}^T) p_N) \leq \frac{c_L}{4(N+1)} (\hat{p}(\frac{2i+3}{2(N+1)}) - \hat{p}(\frac{2i+1}{2(N+1)}))^2.$$

By the boundedness of $f_N'(x)$,

$$g(\hat{p}(\frac{2i+1}{2(N+1)})) + g(\hat{p}(\frac{2i+3}{2(N+1)})) - 2g(\hat{p}(\frac{2i+2}{2(N+1)})) \leq \frac{K_1^2 c_L}{(N+1)^2}.$$

Writing the limiting cost as an integral, and switching to the sub-sequence n defined above,

$$n^2 \sum_{i \in X^n \setminus \{n\}} \{g(e_i^T p_n) + g(e_{i+1}^T p_n) - 2g(\frac{1}{2}(e_i^T + e_{i+1}^T) p_n)\} =$$
$$\frac{n^3}{n+1} \int_0^1 \{g(\hat{p}_n(\frac{2\lfloor nx \rfloor + 1}{2(n+1)})) + g(\hat{p}_n(\frac{2\lfloor nx \rfloor + 3}{2(n+1)})) - 2g(\hat{p}_n(\frac{2\lfloor nx \rfloor + 2}{2(n+1)}))\} dx.$$

By the bound above,

$$\frac{n^3}{n+1} \int_0^1 \{g(\hat{p}_n(\frac{2\lfloor nx \rfloor + 1}{2(n+1)})) + g(\hat{p}_n(\frac{2\lfloor nx \rfloor + 3}{2(n+1)})) - 2g(\hat{p}_n(\frac{2\lfloor nx \rfloor + 2}{2(n+1)}))\} dx \leq$$
$$\frac{n^3}{(n+1)^3} \int_0^1 K_1^2 c_L dx.$$

Applying the dominated convergence theorem,

$$\lim_{n\to\infty} n^2 \sum_{i\in X^n\setminus\{n\}} \{g(e_i^T p_n) + g(e_{i+1}^T p_n) - 2g(\frac{1}{2}(e_i^T + e_{i+1}^T)p_n)\} =$$

$$\int_0^1 \lim_{n\to\infty} \frac{n^3}{n+1}\{g(\hat{p}_n(\frac{2\lfloor nx\rfloor+1}{2(n+1)})) + g(\hat{p}_n(\frac{2\lfloor nx\rfloor+3}{2(n+1)})) - 2g(\hat{p}_n(\frac{2\lfloor nx\rfloor+2}{2(n+1)}))\}dx.$$

By the Taylor expansion above,

$$\lim_{n\to\infty} \frac{n^3}{n+1}\{g(\hat{p}_n(\frac{2\lfloor nx\rfloor+1}{2(n+1)})) + g(\hat{p}_n(\frac{2\lfloor nx\rfloor+3}{2(n+1)})) - 2g(\hat{p}_n(\frac{2\lfloor nx\rfloor+2}{2(n+1)}))\} =$$

$$\lim_{n\to\infty} \frac{1}{8}\frac{n^3}{n+1}\{g''(\cdot) + g''(\cdot)\}(\hat{p}_n(\frac{2\lfloor nx\rfloor+3}{2(n+1)}) - \hat{p}_n(\frac{2\lfloor nx\rfloor+1}{2(n+1)}))^2.$$

By definition,

$$(n+1)(\hat{p}_n(\frac{2\lfloor nx\rfloor+3}{2(n+1)}) - \hat{p}_n(\frac{2\lfloor nx\rfloor+1}{2(n+1)})) = f_n'(\frac{2\lfloor nx\rfloor+2}{2(n+1)})$$

and

$$\lim_{n\to\infty} g''(\hat{p}_n(\frac{2\lfloor nx\rfloor+2}{2(n+1)}) + c_n(\hat{p}_n(\frac{2\lfloor nx\rfloor+3}{2(n+1)}) - \hat{p}_n(\frac{2\lfloor nx\rfloor+2}{2(n+1)}))) = \frac{1}{p(x)},$$

with $c_n \in (0,1)$ for all n, and therefore

$$\lim_{n\to\infty} \frac{n^3}{n+1}\{g(\hat{p}_n(\frac{2\lfloor nx\rfloor+1}{2(n+1)})) + g(\hat{p}_n(\frac{2\lfloor nx\rfloor+3}{2(n+1)})) - 2g(\hat{p}_n(\frac{2\lfloor nx\rfloor+2}{2(n+1)}))\} =$$

$$\lim_{n\to\infty} \frac{1}{4}\frac{(p'(x))^2}{p(x)},$$

proving the second claim.

Turning to the third claim, recall that, by definition,

$$e_i^T u_{a,N} = \frac{\int_{\frac{i}{N+1}}^{\frac{i+1}{N+1}} u_a(x) f(x) dx}{\int_{\frac{i}{N+1}}^{\frac{i+1}{N+1}} f(x) dx.}$$

We define the function, for $x \in [0,1)$, as

$$u_{a,N}(x) = e_{\lfloor (N+1)x \rfloor}^T u_{a,N},$$

and let $u_{a,N}(1) = e_N^T u_{a,N}$. We also define the function

$$\tilde{x}(x) = \frac{2\lfloor (N+1)x \rfloor + 1}{2(N+1)}.$$

By construction, $\hat{p}_N(\tilde{x}(x)) = (N+1) e_{\lfloor (N+1)x \rfloor}^T p_{a,N}$ for all $x \in [0,1)$, and equals $e_N^T p_{a,N}$ for $x = 1$. Therefore,

$$\begin{aligned} u_{a,N}^T p_N &= \sum_{i \in X^N} (e_i^T u_{a,N})(e_i^T p_N) \\ &= \int_0^1 \hat{p}_N(\tilde{x}(x)) u_{a,N}(x) dx. \end{aligned}$$

By the measurability of $u_a(x)$,

$$\lim_{N \to \infty} u_{a,N}(x) = u_a(x).$$

Therefore, by the boundedness of utilities and the dominated convergence theorem,

$$\lim_{n \to \infty} u_{a,n}^T p_n = \int_0^1 p(x) u_a(x) dx.$$

Finally, suppose that, for all N

$$e_i^T p_{a,N} = \int_{\frac{i}{N+1}}^{\frac{i+1}{N+1}} \tilde{p}(x)dx.$$

It follows that $\lim_{n\to\infty} \hat{p}_{a,N}(x) = \tilde{p}(x)$ for all $x \in X$, and therefore $\tilde{p}(x) = p(x)$. □

Lemma 13. *Let $\pi_N(a) \in \mathscr{P}(A)$ and $\{q_{a,N} \in \mathscr{P}(X^N)\}_{a\in A}$ denote optimal policies in the discrete state setting described in section 5.2. For each $a \in A$, the sequence $\{q_{a,N}\}$ satisfies the convergence condition (Definition 1).*

Proof. We begin by noting that the conditions given for the function $f(x)$ satisfy the conditions of Lemma 11, and therefore the sequence q_N satisfies the convergence condition. We will use the constants c_H and c_L to refer to its bounds,

$$\frac{c_H}{N+1} \geq e_i^T q_N \geq \frac{c_L}{N+1},$$

and the constants K_1 and K to refer to the constants described by convergence condition and Lemma 11 for the sequence q_N. By the convention that $q_{a,N} = q_N$ if $\pi_N(a) = 0$, $q_{a,N}$ also satisfies the convergence condition whenever $\pi_N(a) = 0$.

The problem of size N is

$$V_N(q_N;\bar{\theta}) = \max_{\pi_N \in \mathscr{P}(A), \{q_{a,N} \in \mathscr{P}(X^N)\}_{a\in A}} \sum_{a\in A} \pi_N(a)(u_{a,N}^T \cdot q_{a,N}) - \bar{\theta} \sum_{a\in A} \pi_N(a) D_N(q_{a,N}||q_N)$$

subject to

$$\sum_{a\in A} \pi_N(a) q_{a,N} = q_N.$$

Let u_n denote that $|X^N| \times |A|$ matrix whose columns are $u_{a,N}$. Using Lemma 5, we can rewrite the problem as

$$
\begin{aligned}
V_N(q_N;\bar{\theta}) = &\max_{\{p_{x,N}\in\mathscr{P}(A)\}_{i\in X}} \sum_{a\in A} e_a^T pD(q)u_N e_a \\
&- \bar{\theta}N^2 \sum_{i=0}^{N-1} (e_i^T q_N) D_{KL}(p_{i,N}|| \frac{p_{i,N}(e_i^T q_N) + p_{i+1,N}(e_{i+1}^T q_N)}{(e_i^T + e_{i+1}^T) q_N}) \\
&- \bar{\theta}N^2 \sum_{i=1}^{N} (e_i^T q_N) D_{KL}(p_{i,N}|| \frac{p_{i,N}(e_i^T q_N) + p_{i-1,N}(e_{i-1}^T q_N)}{(e_i^T + e_{i-1}^T) q_N}) \\
&- \bar{\theta}N^{-1} \sum_{i=0}^{N-1} (e_i^T q_N) D_{KL}(p_{i,N}|| p_N q_N).
\end{aligned}
$$

The FOC for this problem is, for all $i \in [1, N-1]$ and $a \in A$ such that $e_a^T p_{i,N} > 0$,

$$
\begin{aligned}
e_i^T u_{a,N} - \bar{\theta}N^2 \ln(\frac{e_a^T p_{i,N}(e_i^T + e_{i+1}^T) q_N}{e_a^T (p_{i,N}(e_i^T q_N) + p_{i+1,N}(e_{i+1}^T q_N))}) \\
-\bar{\theta}N^2 \ln(\frac{e_a^T p_{i,N}(e_i^T + e_{i-1}^T) q_N}{e_a^T (p_{i,N}(e_i^T q_N) + p_{i-1,N}(e_{i-1}^T q_N))}) - \bar{\theta}\ln(\frac{e_a^T p_{i,N}}{e_a^T p_N q_N}) - e_i^T \kappa_N = 0,
\end{aligned}
$$

where $\kappa_N \in \mathbb{R}^{N+1}$ are the multipliers (scaled by $e_i^T q_N$) on the constraints that $\sum_{a\in A} e_a^T p_{i,N} = 1$ for all $i \in X$. Defining $q_{-1,N} = q_{N+1,N} = 0$, and defining $p_{-1,N}$ and $p_{N+1,N}$ in arbitrary fashion, we can recover this FOC for all $i \in X$.

Rewriting the FOC in terms of the posteriors, for any a such that $\pi_N(a) > 0$,

$$
\begin{aligned}
e_i^T(u_{a,N} - \kappa_N) &= -\bar{\theta}N^2 \ln(\frac{(e_i^T q_{a,N})(1 + \frac{e_{i+1}^T q_N}{e_i^T q_N})}{(e_{i+1} + e_i)^T q_{a,N}}) - \bar{\theta}N^2 \ln(\frac{(e_i^T q_{a,N})(1 + \frac{e_{i-1}^T q_N}{e_i^T q_N})}{(e_{i-1} + e_i)^T q_{a,N}}) - \bar{\theta}\ln N^{-1}(\frac{e_a^T p_{i,N}}{e_a^T p_N q_N}) \\
&= \bar{\theta}N^2 \ln(1 + \frac{e_{i+1}^T q_{a,N}}{e_i^T q_{a,N}}) - \bar{\theta}N^2 \ln(1 + \frac{e_{i+1}^T q_N}{e_i^T q_N}) + \bar{\theta}N^2 \ln(1 + \frac{e_{i-1}^T q_{a,N}}{e_i^T q_{a,N}}) \\
&- \bar{\theta}N^2 \ln(1 + \frac{e_{i-1}^T q_N}{e_i^T q_N}) - \bar{\theta}\ln N^{-1}(\frac{e_i^T q_{a,N}}{e_i^T q_N}) \\
&= \bar{\theta}N^2 (\ln(\frac{1}{2}(e_{i+1}^T + e_i^T) q_{a,N}) + \ln(\frac{1}{2}(e_{i-1}^T + e_i^T) q_{a,N}) - (2 + N^{-3}) \ln(e_i^T q_{a,N}) + 2\ln 2) \\
&- \bar{\theta}N^2 (\ln(\frac{1}{2}(e_{i+1}^T + e_i^T) q_N) + \ln(\frac{1}{2}(e_{i-1}^T + e_i^T) q_N) - (2 + N^{-3}) \ln(e_i^T q_N) + 2\ln 2).
\end{aligned}
$$

Using Lemma 11, for all $i \in X^N \setminus \{0,N\}$,

$$N^2|\ln(\frac{1}{2}(e_{i+1}^T + e_i^T)q_N) + \ln(\frac{1}{2}(e_{i-1}^T + e_i^T)q_N) - 2\ln(e_i^T q_N)| \leq K.$$

By the boundedness of the utility function,

$$e_i^T \kappa_N \geq -\bar{u} - \bar{\theta}K + \bar{\theta}N^2(\ln(\frac{e_i^T q_{a,N}}{\frac{1}{2}(e_{i+1}^T + e_i^T)q_{a,N}}) + \ln(\frac{e_i^T q_{a,N}}{\frac{1}{2}(e_{i-1}^T + e_i^T)q_{a,N}})) + \bar{\theta}N^{-1}\ln(\frac{e_i^T q_{a,N}}{e_i^T q_N}).$$

By the concavity of the log function,

$$\ln(\frac{1}{2}(e_{i+1}^T + e_i^T)q_{a,N}) + \ln(\frac{1}{2}(e_{i-1}^T + e_i^T)q_{a,N}) + N^{-3}\ln(e_i^T q_N) \leq$$
$$(2+N^{-3})\ln(\frac{1}{2(2+N^{-3})}(e_{i+1}^T + e_{i-1}^T + 2e_i^T)q_{a,N} + \frac{N^{-3}}{2+N^{-3}}e_i^T q_N).$$

and therefore

$$\ln(\frac{1}{2}(e_{i+1}^T + e_i^T)q_{a,N}) + \ln(\frac{1}{2}(e_{i-1}^T + e_i^T)q_{a,N}) + N^{-3}\ln(e_i^T q_N) - (2+N^{-3})\ln(e_i^T q_{a,N})$$
$$\leq (2+N^{-3})\ln(\frac{\frac{1}{2(2+N^{-3})}(e_{i+1}^T + e_{i-1}^T + 2e_i^T)q_{a,N} + \frac{N^{-3}}{2+N^{-3}}e_i^T q_N}{e_i^T q_{a,N}}).$$

It follows that

$$e_i^T \kappa_N \geq -\bar{u} - \bar{\theta}K - (2+N^{-3})\bar{\theta}N^2\ln(\frac{\frac{1}{2(2+N^{-3})}(e_{i+1}^T + e_{i-1}^T + 2e_i^T)q_{a,N} + \frac{N^{-3}}{2+N^{-3}}e_i^T q_N}{e_i^T q_{a,N}}).$$

Exponentiating,

$$(e_i^T q_{a,N}) \exp(-\frac{1}{2+N^{-3}} \bar{\theta}^{-1} N^{-2} (\bar{u} + \bar{\theta} K + e_i^T \kappa_N)) \leq \\ \frac{1}{2(2+N^{-3})} (e_{i+1}^T + e_{i-1}^T + 2e_i^T) q_{a,N} + \frac{N^{-3}}{2+N^{-3}} e_i^T q_N. \quad (31)$$

Summing over a, weighted by $\pi_N(a)$,

$$(e_i^T q_N) \exp(-\frac{1}{2+N^{-3}} \bar{\theta}^{-1} N^{-2} (\bar{u} + \bar{\theta} K + e_i^T \kappa_N)) \leq \\ \frac{1}{2(2+N^{-3})} (e_{i+1}^T + e_{i-1}^T + 2e_i^T) q_N + \frac{N^{-3}}{2+N^{-3}} e_i^T q_N.$$

Taking logs,

$$-\frac{1}{2+N^{-3}} \bar{\theta}^{-1} N^{-2} (\bar{u} + \bar{\theta} K + e_i^T \kappa_N) \leq \ln(\frac{\frac{1}{2(2+N^{-3})} (e_{i+1}^T + e_{i-1}^T + 2e_i^T) q_N + \frac{N^{-3}}{2+N^{-3}} e_i^T q_N}{(e_i^T q_N)}) \\ \leq \ln(1 + \frac{N^{-3}}{2+N^{-3}} + \frac{1}{2+N^{-3}} \frac{K_1 N^{-3}}{c_L N^{-1}}),$$

where the last step follows by Lemma 11, recalling that c_L is the lower bound on $f(x)$. We have

$$\begin{aligned} e_i^T \kappa_N &\geq -2\bar{\theta} N^2 \ln(1 + \frac{N^{-3}}{2+N^{-3}} + \frac{1}{2+N^{-3}} \frac{K_1}{c_L} N^{-2}) - \bar{u} - \bar{\theta} K \\ &\geq -\bar{u} - \bar{\theta} K - \frac{N^{-1}}{2+N^{-3}} - \frac{1}{2+N^{-3}} \frac{K_1}{c_L} \\ &\geq -\bar{u} - \bar{\theta} K - \frac{1}{2} - \frac{1}{2} \frac{K_1}{c_L}. \end{aligned}$$

where the second step follows by the inequality $\ln(1+x) < x$ for $x > 0$.

Turning to the end points, the FOC can be simplified to

$$e_0^T(u_{a,N}-\kappa_N)=\bar{\theta}N^2(\ln(\frac{1}{2}(e_1^T+e_0^T)q_{a,N})-\ln(e_0^Tq_{a,N}))$$
$$-\bar{\theta}N^2(\ln(\frac{1}{2}(e_1^T+e_0^T)q_N)-\ln(e_0^Tq_N))-\bar{\theta}N^{-1}\ln(\frac{e_0^Tq_{a,N}}{e_0^Tq_N}).$$

By the concavity of the log function,

$$\ln(\frac{1}{2}(e_1^T+e_0^T)q_{a,N})+N^{-3}\ln(e_0^Tq_N)-(1+N^{-3})\ln(e_0^Tq_{a,N})$$
$$\leq(1+N^{-3})\ln(\frac{\frac{1}{(1+N^{-3})}\frac{1}{2}(e_1^T+e_0^T)q_{a,N}+\frac{N^{-3}}{1+N^{-3}}e_0^Tq_N}{e_0^Tq_{a,N}}). \quad (32)$$

Therefore,

$$\bar{\theta}n^2\ln(\frac{\frac{1}{2}(e_1^T+e_0^T)q_{a,N}}{e_0^Tq_{a,N}})+\bar{\theta}n^{-1}\ln(\frac{e_0^Tq_N}{e_0^Tq_{a,N}})-\bar{\theta}K$$
$$\leq e_0^T(u_{a,N}-\kappa_N)+\bar{\theta}N^2(\ln(\frac{1}{2}(e_1^T+e_0^T)q_N)-\ln(e_0^Tq_N))$$
$$\leq\bar{\theta}(1+N^{-3})\ln(\frac{\frac{1}{(1+N^{-3})}\frac{1}{2}(e_1^T+e_0^T)q_{a,N}+\frac{N^{-3}}{1+N^{-3}}e_0^Tq_N}{e_0^Tq_{a,N}})+\bar{\theta}K.$$

By the boundedness of the utility function,

$$-\bar{\theta}(1+N^{-3})\ln(\frac{\frac{1}{(1+N^{-3})}\frac{1}{2}(e_1^T+e_0^T)q_{a,N}+\frac{N^{-3}}{1+N^{-3}}e_0^Tq_N}{e_0^Tq_{a,N}})-\bar{u}$$
$$\leq e_0^T\kappa_N+\bar{\theta}N^2\ln(\frac{e_0^Tq_N}{\frac{1}{2}(e_1^T+e_0^T)q_N})$$
$$\leq-\bar{\theta}N^2\ln(\frac{\frac{1}{2}(e_1^T+e_0^T)q_{a,N}}{e_0^Tq_{a,N}})+\bar{\theta}N^{-1}\ln(\frac{e_0^Tq_{a,N}}{e_0^Tq_N})+\bar{u}.$$

By the inequality $\ln(x) \leq x-1$,

$$\bar{\theta}N^{-1}\ln(\frac{e_0^T q_{a,N}}{e_0^T q_N}) \leq \bar{\theta}N^{-1}(\frac{e_0^T q_{a,N}}{e_0^T q_N} - 1)$$
$$\leq \bar{\theta}c_L^{-1},$$

where the latter follows from $e_0^T q_N \geq c_L N^{-1}$. Exponentiating,

$$(e_0^T q_{a,N})\exp(-\bar{\theta}^{-1}(1+N^{-3})^{-1}N^{-2}\bar{\mu}) \leq$$
$$(\frac{1}{(1+N^{-3})}\frac{1}{2}(e_1^T+e_0^T)q_{a,N} + \frac{N^{-3}}{1+N^{-3}}e_0^T q_N)\exp(\bar{\theta}^{-1}(1+N^{-3})^{-1}N^{-2}e_0^T\kappa_N)\frac{e_0^T q_N}{\frac{1}{2}(e_1^T+e_0^T)q_N}$$

and

$$(\frac{1}{2}(e_1^T+e_0^T)q_{a,N})\exp(\bar{\theta}^{-1}N^{-2}e_0^T\kappa_N)\frac{e_0^T q_N}{\frac{1}{2}(e_1^T+e_0^T)q_N} \leq (e_0^T q_{a,N})\exp(\bar{\theta}^{-1}N^{-2}(\bar{\mu}+\bar{\theta}c_L^{-1})).$$

Summing over a, weighted by $\pi_N(a)$,

$$(e_0^T q_N)\exp(-\bar{\theta}^{-1}(1+N^{-3})^{-1}N^{-2}\bar{\mu}) \leq$$
$$(\frac{1}{(1+N^{-3})}\frac{1}{2}(e_1^T+e_0^T)q_N + \frac{N^{-3}}{1+N^{-3}}e_0^T q_N)\exp(\bar{\theta}^{-1}(1+N^{-3})^{-1}N^{-2}e_0^T\kappa_N)\frac{e_0^T q_N}{\frac{1}{2}(e_1^T+e_0^T)q_N},$$

$$(\frac{1}{2}(e_1^T+e_0^T)q_N)\exp(\bar{\theta}^{-1}N^{-2}e_0^T\kappa_N)\frac{e_0^T q_N}{\frac{1}{2}(e_1^T+e_0^T)q_N} \leq (e_0^T q_N)\exp(\bar{\theta}^{-1}N^{-2}(\bar{\mu}+\bar{\theta}c_L^{-1})).$$

Taking logs,

$$-\bar{\theta}N^2(1+N^{-3})(\ln(\frac{\frac{1}{(1+N^{-3})}\frac{1}{2}(e_1^T+e_0^T)q_N + \frac{N^{-3}}{1+N^{-3}}e_0^T q_N}{\frac{1}{2}(e_1^T+e_0^T)q_N}) - \bar{u} \leq e_0^T\kappa_N \leq \bar{u} + \bar{\theta}c_L^{-1}.$$

We can write

$$\ln\left(\frac{\frac{1}{(1+N^{-3})}\frac{1}{2}(e_1^T+e_0^T)q_N+\frac{N^{-3}}{1+N^{-3}}e_0^Tq_N}{\frac{1}{2}(e_1^T+e_0^T)q_N}\right)=\ln\left(\frac{1}{1+N^{-3}}+\frac{\frac{N^{-3}}{1+N^{-3}}e_0^Tq_N}{\frac{1}{2}(e_1^T+e_0^T)q_N}\right)$$
$$\leq\frac{1}{1+N^{-3}}+\frac{2N^{-3}}{1+N^{-3}}-1.$$

Therefore,

$$-\bar{\theta}N^2(1+N^{-3})(\ln(\frac{\frac{1}{(1+N^{-3})}\frac{1}{2}(e_1^T+e_0^T)q_N+\frac{N^{-3}}{1+N^{-3}}e_0^Tq_N}{\frac{1}{2}(e_1^T+e_0^T)q_N})\geq-\bar{\theta}N^{-1}\geq-\bar{\theta}.$$

By Lemma 11,

$$-\bar{\theta}-\bar{u}\leq e_0^T\kappa_N\leq\bar{u}+\bar{\theta}c_L^{-1}.$$

A similar argument applies to the other end-point ($e_N^T\kappa_N$). Summarizing, $e_i^T\kappa_N\geq-B_L$ for some constant $B_L>0$, and $e_i^T\kappa_N\leq B_H$ for some $B_H>0$ if $i\in\{0,N\}$.

Returning to the FOC, for all $i\in X^N\setminus\{0,N\}$,

$$e_i^T\kappa_N\leq\bar{u}+\bar{\theta}K+\bar{\theta}N^2(\ln(\frac{e_i^Tq_{a,N}}{\frac{1}{2}(e_{i+1}^T+e_i^T)q_{a,N}})+\ln(\frac{e_i^Tq_{a,N}}{\frac{1}{2}(e_{i-1}^T+e_i^T)q_{a,N}}))+\bar{\theta}N^{-1}\ln(\frac{e_i^Tq_{a,N}}{e_i^Tq_N}),$$

and as argued above,

$$\bar{\theta}N^{-1}\ln(\frac{e_i^Tq_{a,N}}{e_i^Tq_N})\leq\bar{\theta}c_L^{-1}.$$

Using this bound,

$$\bar{\theta}N^2(\ln(\frac{e_i^Tq_{a,N}}{\frac{1}{2}(e_{i+1}^T+e_i^T)q_{a,N}})+\ln(\frac{e_i^Tq_{a,N}}{\frac{1}{2}(e_{i-1}^T+e_i^T)q_{a,N}}))\geq-(\bar{u}+\bar{\theta}K+B_L+\bar{\theta}c_L^{-1}).$$

For the end-points, the FOC requires that

$$e_0^T \kappa_N \leq \bar{u} - \bar{\theta} N^2 \ln(\frac{e_0^T q_N}{\frac{1}{2}(e_1^T + e_0^T) q_N}) + \bar{\theta} N^2 \ln(\frac{e_0^T q_{a,N}}{\frac{1}{2}(e_1^T + e_0^T) q_{a,N}}) + \bar{\theta} N^{-1} \ln(\frac{e_0^T q_{a,N}}{e_0^T q_N})$$

and

$$e_N^T \kappa_N \leq \bar{u} - \bar{\theta} N^2 \ln(\frac{e_N^T q_{a,N}}{\frac{1}{2}(e_N^T + e_{N-1}^T) q_{a,N}}) + \bar{\theta} N^2 \ln(\frac{e_N^T q_{a,N}}{\frac{1}{2}(e_N^T + e_{N-1}^T) q_{a,N}}) + \bar{\theta} N^{-1} \ln(\frac{e_N^T q_{a,N}}{e_N^T q_N}).$$

Using Lemma 11, we can rewrite these inequalities as

$$\begin{aligned}\bar{\theta} N \ln(\frac{e_N^T q_{a,N}}{\frac{1}{2}(e_N^T + e_{N-1}^T) q_{a,N}}) &\geq -N^{-1}(\bar{u} + B_L + \bar{\theta} c_L^{-1}) + \bar{\theta} N \ln(\frac{e_N^T q_{a,N}}{\frac{1}{2}(e_N^T + e_{N-1}^T) q_{a,N}}) \\ &\geq -N^{-1}(\bar{u} + B_L + \bar{\theta} c_L^{-1}) - \bar{\theta} K \\ &\geq -(\bar{u} + \bar{\theta} K + B_L + \bar{\theta} c_L^{-1}),\end{aligned}$$

and likewise

$$\bar{\theta} N \ln(\frac{e_0^T q_{a,N}}{\frac{1}{2}(e_1^T + e_0^T) q_{a,N}}) \geq -(\bar{u} + \bar{\theta} K + B_L + \bar{\theta} c_L^{-1}).$$

Define $\hat{q}_{a,N}(x)$ as in Lemma 12. Define the function

$$l_{a,N}(x) = (N+1)(\ln(\hat{q}_{a,N}(x)) - \ln(\hat{q}_{a,N}(x - \frac{1}{2(N+1)})))$$

for any $x \in [\frac{1}{2(N+1)}, 1]$. For any $i \in X^N \setminus \{0\}$,

$$l_{a,N}(\frac{2i+1}{2(N+1)}) = (N+1) \ln(\frac{(N+1) e_i^T q_{a,N}}{\frac{1}{2}(N+1)(e_i^T + e_{i-1}^T) q_{a,N}}),$$

and for any $i \in X^N \setminus \{N\}$,

$$l_{a,N}(\frac{2i+2}{2(N+1)}) = (N+1)\ln(\frac{\frac{1}{2}(N+1)(e_i^T + e_{i+1}^T)q_{a,N}}{(N+1)e_i^T q_{a,N}}).$$

Therefore, for any $i \in X^N \setminus \{0,N\}$, the lower bound can be written as

$$\bar{\theta}\frac{N^2}{N+1}(l_{a,N}(\frac{2i+2}{2(N+1)}) - l_{a,N}(\frac{2i+1}{2(N+1)})) \leq (\bar{u} + \bar{\theta}K + B_L + \bar{\theta}c_L^{-1}).$$

The lower endpoint bound is

$$\bar{\theta}\frac{N}{N+1}l_{a,N}(\frac{1}{(N+1)}) \leq (\bar{u} + \bar{\theta}K + B_L + \bar{\theta}c_L^{-1}).$$

The upper endpoint bound is

$$\bar{\theta}\frac{N}{N+1}l_{a,N}(1) \geq -(\bar{u} + \bar{\theta}K + B_L + \bar{\theta}c_L^{-1}).$$

We also have, for all $i \in X^N \setminus \{N\}$

$$\begin{aligned} &\bar{\theta}\frac{N^2}{N+1}(l_{a,N}(\frac{2i+3}{2(N+1)}) - l_{a,N}(\frac{2i+2}{2(N+1)})) \\ &= \bar{\theta}N^2(\ln(\frac{(N+1)e_{i+1}^T q_{a,N}}{\frac{1}{2}(N+1)(e_{i+1}^T + e_i^T)q_{a,N}}) - \ln(\frac{\frac{1}{2}(N+1)(e_i^T + e_{i+1}^T)q_{a,N}}{(N+1)e_i^T q_{a,N}})) \\ &\leq 0, \end{aligned}$$

by the concavity of the log function. Therefore, for all $j \in \{2,3,\ldots,2(N+1)\}$

$$\bar{\theta}\frac{N^2}{N+1}(l_{a,N}(\frac{j+1}{2(N+1)}) - l_{a,N}(\frac{j}{2(N+1)})) \leq (\bar{u} + \bar{\theta}K + B_L + \bar{\theta}c_L^{-1}).$$

It follows that, for all $j \in \{2,3,\ldots,2(N+1)\}$

$$\begin{aligned} l_{a,N}(\frac{j}{2(N+1)}) &= l_{a,N}(\frac{2}{2(N+1)}) + \sum_{k=2}^{j-1}(l_{a,N}(\frac{k+1}{2(N+1)}) - l_{a,N}(\frac{k}{2(N+1)})) \\ &\leq \bar{\theta}^{-1}(\bar{u} + \bar{\theta}K + B_L + \bar{\theta}c_L^{-1})\frac{N+1}{N}(1+\frac{j-2}{N}). \end{aligned}$$

Similarly, for all $j \in \{2,3,\ldots,2(N+1)\}$,

$$l_{a,N}(1) = l_{a,N}(\frac{j}{2(N+1)}) + \sum_{k=j-1}^{2N}(l_{a,N}(\frac{k+1}{2(N+1)}) - l_{a,n}(\frac{k}{2(N+1)}))$$

and therefore

$$\begin{aligned} -l_{a,N}(\frac{j}{2(N+1)}) &= -l_{a,n}(1) + \sum_{k=j-1}^{2N}(l_{a,N}(\frac{k+1}{2(N+1)}) - l_{a,n}(\frac{k}{2(N+1)})) \\ &\leq \bar{\theta}^{-1}(\bar{u} + \bar{\theta}K + B_L + \bar{\theta}c_L^{-1})\frac{N+1}{N}(1+\frac{2(N+1)-j}{N^2}). \end{aligned}$$

It follows that

$$\begin{aligned} |l_{a,N}(\frac{j}{2(N+1)})| &\leq 2\bar{\theta}^{-1}(\bar{u} + \bar{\theta}K + B_L + \bar{\theta}c_L^{-1})\frac{N+1}{N} \\ &\leq 4\bar{\theta}^{-1}(\bar{u} + \bar{\theta}K + B_L + \bar{\theta}c_L^{-1}). \end{aligned}$$

Note that there must exist some $\tilde{i}_{a,N} \in X^N$ such that $e_{\tilde{i}_{a,N}}^T q_{a,N} \geq \frac{1}{N+1}$, implying that

$$\ln((N+1)e_{\tilde{i}_{a,n}}^T q_{a,N}) \geq 0.$$

By the definition of $l_{a,N}$, for any $i \in X^N \setminus \{0\}$,

$$l_{a,N}(\frac{2i+1}{2(N+1)}) + l_{a,N}(\frac{2i}{2(N+1)}) = (N+1)\ln(\frac{(N+1)e_i^T q_{a,N}}{(N+1)e_{i-1}^T q_{a,N}}).$$

For any $i > \tilde{i}_{a,N}$,

$$\begin{aligned}
\ln((N+1)e_i^T q_{a,N}) &= \ln((N+1)e_{\tilde{i}_{a,n}}^T q_{a,N}) + \sum_{j=\tilde{i}_{a,n}+1}^{i} \ln(\frac{(N+1)e_j^T q_{a,N}}{(N+1)e_{j-1}^T q_{a,N}}) \\
&= \ln((N+1)e_{\tilde{i}_{a,n}}^T q_{a,N}) + \frac{1}{N+1} \sum_{j=\tilde{i}_{a,n}+1}^{i} l_{a,N}(\frac{2j+1}{2(N+1)}) + l_{a,N}(\frac{2j}{2(N+1)}) \\
&\geq -\frac{1}{N+1} \sum_{j=\tilde{i}_{a,n}+1}^{i} 8\bar{\theta}^{-1}(\bar{u} + \bar{\theta}K + B_L + \bar{\theta}c_L^{-1}) \\
&\geq -8\bar{\theta}^{-1}(\bar{u} + \bar{\theta}K + B_L + \bar{\theta}c_L^{-1}).
\end{aligned}$$

Similarly, for any $i < \tilde{i}_{a,N}$,

$$\ln((N+1)e_{\tilde{i}_{a,n}}^T q_{a,N}) = \ln((N+1)e_i^T q_{a,N}) + \sum_{j=i+1}^{\tilde{i}_{a,n}} \ln(\frac{(N+1)e_{j+1}^T q_{a,N}}{(N+1)e_j^T q_{a,N}}).$$

Therefore,

$$\begin{aligned}
\ln((N+1)e_i^T q_{a,N}) &\geq -\sum_{j=i+1}^{\tilde{i}_{a,n}} \ln(\frac{(N+1)e_{j+1}^T q_{a,N}}{(N+1)e_j^T q_{a,N}}) \\
&\geq -8\bar{\theta}^{-1}(\bar{u} + \bar{\theta}K + B_L + \bar{\theta}c_L^{-1}).
\end{aligned}$$

Repeating this argument, there must be some $\hat{i}_{a,N}$ such that $e_{\hat{i}_{a,N}}^T q_{a,N} \leq N^{-1}$, and using the bounds on $l_{a,N}$ in similar fashion yields

$$\ln((N+1)e_i^T q_{a,N}) \leq 8\bar{\theta}^{-1}(\bar{u} + \bar{\theta}K + B_L + \bar{\theta}c_L^{-1}).$$

It follows that there exists a constant $c \in (0,1)$ such that, for all N, $a \in A$ such that $\pi_N(a) > 0$, and $i \in X^N$,

$$\frac{c^{-1}}{(N+1)} \geq e_i^T q_{a,N} \geq \frac{c}{N+1},$$

demonstrating that $q_{a,N}$ satisfies the first part of the convergence condition.

Using the bound on $l_{a,N}$, and a Taylor expansion, for some $a \in (0,1)$

$$|(N+1)\ln(\frac{\frac{1}{2}(N+1)(e_i^T + e_{i+1}^T)q_{a,N}}{(N+1)e_i^T q_{a,N}})| = \frac{(N+1)|\frac{1}{2}(e_{i+1}^T - e_i^T)q_{a,N}|}{e_i^T q_{a,N} + \frac{a}{2}(e_{i+1}^T - e_i^T)q_{a,N}}$$
$$\leq 4\bar{\theta}^{-1}(\bar{u} + \bar{\theta}K + B_L + \bar{\theta}c_L^{-1}),$$

and therefore, by the bound on $e_i^T q_{a,N}$,

$$(N+1)^2|\frac{1}{2}(e_{i+1}^T - e_i^T)q_{a,N}| \leq B$$

for some $B > 0$. By a similar argument,

$$(N+1)^2|\frac{1}{2}(e_{i+1}^T - e_{i-1}^T)q_{a,N}| \leq 4B.$$

Returning to the first-order condition, for $i \in X^N \setminus \{0,N\}$, and using some of the bounds employed previously,

$$e_i^T \kappa_N \leq \bar{u} + \bar{\theta}K + \bar{\theta}c_L + \bar{\theta}N^2(\ln(\frac{e_i^T q_{a,N}}{\frac{1}{2}(e_{i+1}^T + e_i^T)q_{a,N}}) + \ln(\frac{e_i^T q_{a,N}}{\frac{1}{2}(e_{i-1}^T + e_i^T)q_{a,N}})).$$

By the inequality $\ln(x) \leq x - 1$,

$$e_i^T \kappa_N \leq \bar{u} + \bar{\theta}K + \bar{\theta}c_L + \bar{\theta}N^2(\frac{\frac{1}{2}(e_i^T - e_{i+1}^T)q_{a,N}}{\frac{1}{2}(e_{i+1}^T + e_i^T)q_{a,N}} + \frac{\frac{1}{2}(e_i^T - e_{i-1}^T)q_{a,N}}{\frac{1}{2}(e_{i-1}^T + e_i^T)q_{a,N}})$$

Multiplying through,

$$\begin{aligned}
&\frac{1}{2}(e_{i-1}^T + e_i^T)q_{a,N}(e_i^T\kappa_N - \bar{u} - \bar{\theta}K - \bar{\theta}c_L) \\
&\leq \bar{\theta}N^2(\frac{1}{2}(e_i^T - e_{i+1}^T)q_{a,N} + \frac{1}{2}(e_i^T - e_{i-1}^T)q_{a,N}\frac{\frac{1}{2}(e_{i+1}^T + e_i^T)q_{a,N}}{\frac{1}{2}(e_{i-1}^T + e_i^T)q_{a,N}}). \\
&\leq \bar{\theta}N^2(\frac{1}{2}(2e_i^T - e_{i+1}^T - e_{i-1}^T)q_{a,N} + \frac{1}{2}(e_i^T - e_{i-1}^T)q_{a,N}(\frac{\frac{1}{2}(e_{i+1}^T - e_{i-1}^T)q_{a,N}}{\frac{1}{2}(e_{i-1}^T + e_i^T)q_{a,N}})).
\end{aligned}$$

Using the bounds above,

$$\begin{aligned}
\frac{1}{2}(e_{i-1}^T + e_i^T)q_{a,N}(e_i^T\kappa_N - \bar{u} - \bar{\theta}K - \bar{\theta}c_L) &\leq \bar{\theta}N^2(\frac{1}{2}(2e_i^T - e_{i+1}^T - e_{i-1}^T)q_{a,N} + \frac{B}{(N+1)^2}(\frac{\frac{4B}{(N+1)^2}}{\frac{c}{N+1}})) \\
&\leq \bar{\theta}N^2(\frac{1}{2}(2e_i^T - e_{i+1}^T - e_{i-1}^T)q_{a,N}) + \frac{4B^2N^2}{c(N+1)^3}.
\end{aligned}$$

Therefore,

$$c(e_i^T\kappa_N - \bar{u} - \bar{\theta}K - \bar{\theta}c_L) \leq \bar{\theta}\frac{N+1}{N}N^3(\frac{1}{2}(2e_i^T - e_{i+1}^T - e_{i-1}^T)q_{a,N}) + \frac{4B^2}{c}.$$

Summing over a, weighted by $\pi_N(a)$, and applying Lemma 11,

$$c(e_i^T\kappa_N - \bar{u} - \bar{\theta}K - \bar{\theta}c_L) \leq 2\bar{\theta}K_1 + \frac{4B^2}{c}.$$

Therefore, $|e_i^T\kappa_N|$ is bounded below by some $B_\kappa > 0$ for all $i \in X^N$ (recalling that this was shown for $i \in \{0, N\}$ previously). It also follows the term

$$\begin{aligned}
(N+1)^3(\frac{1}{2}(2e_i^T - e_{i+1}^T - e_{i-1}^T)q_{a,N}) &\geq \frac{(N+1)^2}{N^2}c(e_i^T\kappa_N - \bar{u} - \bar{\theta}K - \bar{\theta}c_L - \frac{4B^2}{c^2}) \\
&\geq -2c(B_\kappa + \bar{u} + \bar{\theta}K + \bar{\theta}c_L + \frac{4B^2}{c^2})
\end{aligned}$$

is bounded below.

Recalling equation (31), and employing the upper bound on $|e_i^T \kappa_N|$,

$$(e_i^T q_{a,N}) \exp(-\frac{1}{2+N^{-3}} \bar{\theta}^{-1} N^{-2} (\bar{u} + \bar{\theta} K + B_\kappa))$$
$$\leq \frac{1}{2(2+N^{-3})} (e_{i+1}^T + e_{i-1}^T + 2e_i^T) q_{a,N} + \frac{N^{-3}}{2+N^{-3}} e_i^T q_N.$$

Rewriting this,

$$(e_i^T q_{a,N}) (\exp(-\frac{1}{2+N^{-3}} \bar{\theta}^{-1} N^{-2} (\bar{u} + \bar{\theta} K + B_\kappa)) - 1)$$
$$\leq \frac{1}{2(2+N^{-3})} (e_{i+1}^T + e_{i-1}^T + 2e_i^T) q_{a,N} + \frac{N^{-3}}{2+N^{-3}} e_i^T (q_N - q_{a,N})$$

By the upper bound on $e_i^T q_N \leq \frac{c_H}{N+1}$ and $e_i^T q_{a,N} \geq \frac{c}{N+1}$,

$$\frac{(N+1)^3}{2} (e_{i+1}^T + e_{i-1}^T - 2e_i^T) q_{a,N} \geq$$
$$(2+N^{-3})(N+1)^2 (\exp(-\frac{1}{2+N^{-3}} \bar{\theta}^{-1} N^{-2} (\bar{u} + \bar{\theta} K + B_\kappa)) - 1) - \frac{c_H - c}{N^3} (N+1)^2.$$

By the inequality $\exp(x) - 1 \geq x$,

$$\frac{(N+1)^3}{2} (e_{i+1}^T + e_{i-1}^T - 2e_i^T) q_{a,N} \geq -\frac{(N+1)^2}{N^2} \bar{\theta}^{-1} (\bar{u} + \bar{\theta} K + B_\kappa) - \frac{c_H - c}{N^3} (N+1)^2$$
$$\geq -2\bar{\theta}^{-1} (\bar{u} + \bar{\theta} K + B_\kappa) - 2c_H + c.$$

Therefore, the first statement in the second part of the convergence condition (Definition 1) is satisfied.

Finally, we consider the endpoints. The first-order condition is

$$\bar{\theta}N^2(\ln(\frac{1}{2}(e_1^T+e_0^T)q_{a,N})-\ln(e_0^T q_{a,N}))=$$
$$e_0^T(u_{a,N}-\kappa_N)+\bar{\theta}N^2(\ln(\frac{1}{2}(e_1^T+e_0^T)q_N)-\ln(e_0^T q_N))+\bar{\theta}N^{-1}\ln(\frac{e_0^T q_{a,N}}{e_0^T q_N}).$$

We can bound this as

$$\begin{aligned}&-N^{-1}(\bar{u}+B_\kappa)-\bar{\theta}K+\bar{\theta}N^{-2}\ln(\frac{c}{c_H})\\ &\leq \bar{\theta}N(\ln(\frac{1}{2}(e_1^T+e_0^T)q_{a,N})-\ln(e_0^T q_{a,N}))\\ &\leq N^{-1}(\bar{u}+B_\kappa+\bar{\theta}c_L^{-1})+\bar{\theta}K,\end{aligned}$$

and note that because $\sum_{i\in X^N} e_i^T q_{a,N}=\sum_{i\in X^N} e_i^T q_N=1$, we must have $c_H\geq c$. Therefore,

$$\bar{\theta}\ln(\frac{c}{c_H})\leq\bar{\theta}N^{-2}\ln(\frac{c}{c_H}).$$

Using a Taylor expansion,

$$\ln(\frac{1}{2}(e_1^T+e_0^T)q_{a,N})-\ln(e_0^T q_{a,N})=\frac{\frac{1}{2}(e_1^T-e_0^T)q_{a,N}}{e_0^T q_{a,N}+\frac{a}{2}(e_1^T+e_0^T)q_{a,N}}$$

for some $a\in(0,1)$. Therefore,

$$N^2|\frac{1}{2}(e_1^T-e_0^T)q_{a,N}|\leq\frac{c}{\bar{\theta}}(\bar{u}+B_\kappa+\bar{\theta}K+\bar{\theta}\max(\ln(\frac{c_H}{c}),c_L^{-1})).$$

A similar logic holds for the other endpoint, and therefore the convergence condition is satisfied. □

A.11 Proof of Theorem 3

By the boundedness of $\mathscr{P}(A)$, there exists a convergent sub-sequence of the optimal policy $\pi_N(a)$, which we denote by n. Define

$$\pi(a) = \lim_{n\to\infty} \pi_n(a).$$

By Lemma 13, for all $a \in A$, each sequence of optimal policies $\{q_{a,N}\}$ satisfies the convergence condition (Definition 1). Therefore, by Lemma 12, each sequence $\{\hat{q}_{a,N}(x)\}$ has a convergent sub-sequence that converges to a differentiable function $f_a^*(x)$, whose derivative is Lipschitz continuous, with full support on $[0,1]$. We can construct a sub-sequence in which $\pi_n(a)$ and all $\{\hat{q}_{a,n}(x)\}$ converge by iteratively applying this argument. Denote this sequence by n.

We can write the discrete value function as, using Lemma 5, as

$$\begin{aligned}
V_N(q_N;\bar{\theta}) = &\max_{\{p_{x,N}\in\mathscr{P}(A)\}_{i\in X}} \sum_{a\in A} e_a^T pD(q)u_N e_a \\
&- \bar{\theta}N^2 \sum_{a\in A}(e_a^T pq)\sum_{i=0}^{N-1}[(e_i^T q_{a,N})\ln(\frac{e_i^T q_{a,N}}{\bar{q}_{i,a,N}}) + (e_{i+1}^T q_{a,N})\ln(\frac{e_{i+1}^T q_{a,N}}{\bar{q}_{i,a,N}})] \\
&+ \bar{\theta}N^2 \sum_{i=0}^{N-1}[(e_i^T q_N)\ln(\frac{e_i^T q_N}{\bar{q}_{i,a,N}}) + (e_{i+1}^T q_N)\ln(\frac{e_{i+1}^T q_N}{\bar{q}_{i,a,N}})] \\
&- \bar{\theta}N^{-1}\sum_{i=0}^{N-1}(e_i^T q_N)D_{KL}(p_{i,N}||p_N q_N).
\end{aligned}$$

We can re-arrange this to

$$
\begin{aligned}
V_N(q_N;\bar{\theta}) = &\max_{\{p_{x,N}\in\mathscr{P}(A)\}_{i\in X}} \sum_{a\in A} e_a^T pD(q)u_N e_a \\
&- \bar{\theta}N^2 \sum_{a\in A}(e_a^T pq)\sum_{i=0}^{N-1}[g(e_i^T q_{a,N}) + g(e_{i+1}^T q_{a,N}) - 2g(\frac{1}{2}(e_i^T + e_{i+1}^T)q_{a,N})] \\
&+ \bar{\theta}N^2 \sum_{i=0}^{N-1}[g(e_i^T q_N) + g(e_{i+1}^T q_N) - 2g(\frac{1}{2}(e_i^T + e_{i+1}^T)q_N)] \\
&- \bar{\theta}N^{-1}\sum_{i=0}^{N-1}(e_i^T q_N)D_{KL}(p_{i,N}||p_N q_N).
\end{aligned}
$$

By Lemma 12 and the boundedness of the KL divergence,

$$
\begin{aligned}
\lim_{n\to\infty} V_n(q_n;\bar{\theta}) = &\sum_{a\in A}\pi(a)\int_0^1 u_a(x)f_a(x)dx \\
&- \frac{\bar{\theta}}{4}\sum_{a\in A}\{\pi(a)\int_0^1 \frac{(f_a'(x))^2}{f_a(x)}dx\} + \frac{\bar{\theta}}{4}\int_0^1 \frac{(f'(x))^2}{f(x)}dx.
\end{aligned}
$$

Suppose that $\pi(a)$ and the $f_a(x)$ functions do not maximize this expression (subject to the constraints stated in Theorem 3). Let $\pi^*(a)$ and $f_a^*(x)$ be maximizers. Define, for all $N\in\mathbb{N}$,

$$\tilde{\pi}_N(a) = \pi^*(a),$$

$$e_i^T \tilde{q}_{a,N} = \int_{\frac{i}{N+1}}^{\frac{i+1}{N+1}} f_a^*(x)dx.$$

Note that, by construction, $\tilde{q}_{a,N}\in\mathscr{P}(X^N)$ and $\sum_{a\in A}\tilde{\pi}_N(a)\tilde{q}_{a,N} = q_N$. That is, the constraints of the discrete-state problem are satisfied for all N. Denote the value function under these policies as $\tilde{V}_N(q_N;\bar{\theta})$.

Because of the constraints stated in Theorem 3, each f_a^* satisfies the conditions of Lemma 11, and therefore the sequence $\tilde{q}_{a,N}$ satisfies the convergence condition for all $a\in A$. It follows by Lemma 12 that this sequence of policies delivers, in the limit, the value function $V(f;\bar{\theta})$. If this function is strictly larger than $\lim_{n\to\infty} V_n(q_n;\bar{\theta})$, there must exist some $\bar{n}$ such that

$$\tilde{V}_{\bar{n}}(q_{\bar{n}};\bar{\theta}) > V_{\bar{n}}(q_{\bar{n}};\bar{\theta}),$$

contradicting optimality. Therefore, the functions $f_a(x)$ and $\pi(a)$ are maximizers.

It remains to show that

$$\lim_{n\to\infty} \sum_{i=0}^{\lfloor xn \rfloor} e_i^T q_{a,n} = \int_0^x f_a(y)dy.$$

Note that

$$e_i^T q_{a,n} = (n+1) \int_{\frac{i}{n+1}}^{\frac{i+1}{n+1}} \hat{q}_{a,n}(\frac{2i+1}{2(n+1)})dy,$$

where $\hat{q}_{a,n}$ is the function defined in Lemma 12. Therefore, the sum is equal to

$$\sum_{i=0}^{\lfloor xn \rfloor} e_i^T q_{a,n} = \int_0^{\frac{\lfloor xn \rfloor+1}{n+1}} \hat{q}_{a,n}(\frac{\lfloor (n+1)y+\frac{1}{2} \rfloor + \frac{1}{2}}{(n+1)})dy.$$

By the boundedness of $\hat{q}_{a,n}$ (which follows from the convergence condition) and the dominated convergence theorem,

$$\lim_{n\to\infty} \int_0^{\frac{\lfloor xn \rfloor+1}{n+1}} \hat{q}(\frac{\lfloor (n+1)y+\frac{1}{2} \rfloor + \frac{1}{2}}{(n+1)})dy = \int_0^x f_a(y)dy,$$

as required.

A.12 Proof of Lemma 7

We begin by observing that any information structure $p \in \mathscr{P}_{LipG}(A)$ defines unconditional action frequencies $\pi \in \mathscr{P}(A)$ and posteriors $f_a \in \mathscr{P}_{LipG}([0,1])$ satisfying (25), using definitions (26). And conversely, any unconditional action frequencies and posteriors satisfying (25) define an information structure, using definitions (27). Hence the set of candidate structures is the same in both problems, and the problems are equivalent if the two objective functions are equivalent as well. It is also easily seen that in each problem, the first term of the objective function is the expected value of the DM's reward $u(x,a)$, integrating over the joint distribution for (x,a). Hence it remains only to establish that the remaining terms of the objective function are equivalent as well.

Consider any information structure $p \in \mathscr{P}_{LipG}(A)$ and the corresponding unconditional action frequencies and posteriors, and let x be any point at which $f(x) > 0$, and at which $p_a(x)$ is twice differentiable for all a (and as a consequence, $f_a(x)$ is twice differentiable for all a as well). (We note that, given the Lipschitz continuity of the first derivatives, the set of x for which this is true must be of full measure.) Then the fact that $\sum_{a \in A} p_a(x) = 1$ for all x implies that

$$\sum_{a \in A} p_a''(x) = 0, \tag{33}$$

and similarly, constraint (25) implies that

$$\sum_{a \in A} \pi(a) f_a''(x) = f''(x). \tag{34}$$

At any such point, the definition of the Fisher information implies that

$$\begin{aligned}
I^{Fisher}(x) &\equiv \sum_{a\in A} \frac{(p_a'(x))^2}{p_a(x)} \\
&= \sum_a p_a''(x) - \sum_{a\in A} p_a(x) \frac{\partial^2 \log p_a(x)}{\partial x^2} \\
&= -\frac{\pi(a) f_a(x)}{f(x)} \frac{\partial^2}{\partial x^2} [\log \pi(a) + \log f_a(x) - \log f(x)] \\
&= \frac{1}{f(x)} \left[\sum_{a\in A} \pi(a) \frac{(f_a'(x))^2}{f_a(x)} - \sum_{a\in A} \pi(a) f_a''(x) - \frac{(f'(x))^2}{f(x)} + f''(x) \right] \\
&= \frac{1}{f(x)} \left[\sum_{a\in A} \pi(a) \frac{(f_a'(x))^2}{f_a(x)} - \frac{(f'(x))^2}{f(x)} \right].
\end{aligned}$$

Here the first line is the definition of the Fisher information (given in the lemma), and the second line follows from twice differentiating the function $\log p_a(x)$ with respect to x. In the third line, the first term from the second line vanishes because of (33); the remaining term from the second line is rewritten using (27). The fourth line follows from the third line by twice differentiating each of the terms inside the square brackets with respect to x. The fifth line then follows from (34).

Since this result holds for a set of x of full measure, we obtain expression

$$\int_0^1 f(x) I^{Fisher}(x) dx = \sum_{a\in A} \pi(a) \int_0^1 \frac{(f_a'(x))^2}{f_a(x)} dx - \int_0^1 \frac{(f'(x))^2}{f(x)} dx$$

for the mean Fisher information. This shows that the information-cost terms in both objective functions are equivalent, and hence the two problems are equivalent, and have equivalent solutions.

A.13 Proof of Lemma 8

Write the value function in sequence-problem form:

$$W(q_0,\lambda;\Delta) = \max_{\{p_{\Delta j}\},\tau} E_0[\hat{u}(q_\tau) - \kappa\tau)] -$$

$$\lambda E_0[\Delta^{1-\rho} \sum_{j=0}^{\tau\Delta^{-1}} \{\frac{1}{\rho} C(\{p_{\Delta j,x}\}_{x\in X}, q_{\Delta j}(\cdot))^\rho - \Delta^\rho c^\rho\}].$$

Define

$$\bar{u} = \max_{a\in A, x\in X} u(a,x).$$

By the weak positivity of the cost function $C(\cdot)$, it follows that

$$W(q_0,\lambda;\Delta) \leq \bar{u} + \max_{\tau} E_0[-\kappa\tau + \Delta \sum_{j=0}^{\tau\Delta^{-1}-1} \lambda c^\rho].$$

Because $\lambda \in (0, \kappa c^{-\rho})$, the expression

$$-\kappa\tau + \Delta \sum_{j=0}^{\tau\Delta^{-1}-1} \lambda c^\rho = (\lambda c^\rho - \kappa)\tau$$

is weakly negative, and therefore

$$W(q_0,\lambda;\Delta) \leq \bar{u}.$$

By a similar argument, there is a smallest possible decision utility $\underline{\mathrm{u}}$, and because stopping now and deciding is always feasible,

$$W(q_0,\lambda;\Delta) \geq \underline{\mathrm{u}}.$$

Therefore, $W(q_0,\lambda;\Delta)$ is bounded for all $\lambda \in (0, \kappa c^{-\rho})$ and all Δ. Note that this argument also shows that

$$E_0[\tau](\kappa-\lambda c^{\rho}) \leq \bar{u}-W(q_0,\lambda;\Delta),$$

and hence that

$$E_0[\tau] \leq \frac{\bar{u}-\underline{\mathrm{u}}}{(\kappa-\lambda c^{\rho})}.$$

We can define the "state-specific" value function, $W(q_t,\lambda;\Delta,x)$, which is the value function conditional on the true state being x. The state-specific value function has a recursive representation, in the region in which the DM continues to gather information:

$$\begin{aligned} W(q_t,\lambda;\Delta,x) &= -\kappa\Delta+\lambda\Delta^{1-\rho}(\Delta^{\rho}c^{\rho}-\frac{1}{\rho}C(\cdot)^{\rho})+ \\ & \sum_{s\in S:\, e_s^T p_t e_x>0}(e_s^T p_t^* e_x)W(q^*_{t+\Delta,s},\lambda;\Delta,x). \end{aligned}$$

In this equation, we take the optimal information structure as given. Note that, by construction, wherever the DM does not choose to stop, the expected value of the state-specific value functions is equal to the value function.

$$\sum_{x\in X} q_{t,x}W(q_t,\lambda;\Delta,x) = W(q_t,\lambda;\Delta).$$

By the optimality of the policies, we have

$$W(q_t,\lambda;\Delta) \geq \sum_{x\in X} q_{t,x}W(q',\lambda;\Delta,x),$$

for any q' in $\mathscr{P}(X)$. Suppose not; then the DM could simply adopt the information structure associated with beliefs q' and achieve higher utility, contradicting the optimality of the policy.

The convexity of the value function follows from the observation that

$$\begin{aligned} W(\alpha q+(1-\alpha)q',\lambda;\Delta) &= \alpha \sum_{x\in X} q_x W(\alpha q+(1-\alpha)q',\lambda;\Delta,x) + \\ &\quad (1-\alpha)\sum_{x\in X} q'_x W(\alpha q+(1-\alpha)q',\lambda;\Delta,x), \end{aligned}$$

and using the inequality above,

$$W(\alpha q+(1-\alpha)q',\lambda;\Delta) \leq \alpha W(q,\lambda;\Delta)+(1-\alpha)W(q',\lambda;\Delta).$$

A.14 Proof of Lemma 9

Consider an alternative policy that mixes (in the sense of Condition 2) the optimal signal structure and an uninformative signal, with probabilities $1-a$ and a, respectively. We must have

$$-\sum_{s\in S}(e_s^T r_{t,n}^*)(W(q_{t,n,s}^*,\lambda;\Delta_n)-W(q_{t,n},\lambda;\Delta_n))-\lambda\Delta_n^{1-\rho}C(p_{t,n}^*,q_{t,n})^{\rho-1}\frac{\partial C(p_{t,n}(a),q_{t,n})}{\partial a}|_{a=0^+}\leq 0,$$

which is the first-order condition at the optimal policy in the direction of adding a little bit of the uninformative signal (decreasing a). By the convexity of $C(\cdot)$ and Condition 1,

$$C(p_{t,n}^*,q_{t,n})+\frac{\partial C(p_{t,n}(a),q_{t,n})}{\partial a}|_{a=0^+}\leq 0,$$

and therefore we must have

$$\sum_{s\in S}(e_s^T r_{t,n}^*)(W(q_{t,n,s}^*,\lambda;\Delta_n)-W(q_{t,n},\lambda;\Delta_n))\geq \lambda\Delta_n^{1-\rho}C(p_{t,n}^*,q_{t,n})^{\rho}.$$

Applying the Bellman equation in the continuation region,

$$(\kappa - \lambda c^{\rho})\Delta_n + \frac{\lambda}{\rho}\Delta_n^{1-\rho} C(p^*_{t,n}, q_{t,n})^{\rho} \geq \lambda \Delta_n^{1-\rho} C(p^*_{t,n}, q_{t,n})^{\rho}.$$

Therefore,

$$\lambda(1 - \frac{1}{\rho})\Delta_n^{-\rho} C(p^*_{t,n}, q_{t,n})^{\rho} \leq (\kappa - \lambda c^{\rho}).$$

It follows by the assumption that $\lambda \in (0, \kappa c^{-\rho})$ and that $\rho > 1$ that

$$C(p^*_{t,n}, q_{t,n}) \leq \Delta_n (\frac{\theta}{\lambda})^{\frac{1}{\rho-1}},$$

for the constant $\theta = \lambda(\rho \frac{\kappa - \lambda c^{\rho}}{\lambda(\rho-1)})^{\frac{\rho-1}{\rho}} > 0$.

A.15 Proof of Lemma 10

We begin by discussing the convergence of stopping times. We have assumed that

$$E_0[\tau_n] \leq \bar{\tau},$$

for some strictly positive constant $\bar{\tau}$ and all n. It follows by the positivity of τ_n that the laws of τ_n are tight, and therefore there exists a sub-sequence that converges in measure. Pass to this sub-sequence (which we will also index by n), and let τ denote the limit of this sub-sequence.

The beliefs $q_{t,n}$ are a family of $\mathbb{R}^{|X|}$-valued stochastic processes, with $q_{t,n} \in \mathscr{P}(X)$ for all $t \in [0, \infty)$ and $n \in \mathbb{N}$. Construct them as RCLL processes by assuming that $q_{\Delta_n j + \varepsilon, n} = q_{\Delta_n j, n}$ for all m, $\varepsilon \in [0, \Delta_n)$, and $j \in \mathbb{N}$. We next establish that the laws of $q_{t,n}$ are tight. By

Condition 5 and Lemma 9,

$$\frac{m}{2}\sum_{s\in S}(e_s^T p_n(q_{t,n})q_{t,n})||q_{s,n}(q_{t,n})-q_{t,n}||_2^2 \leq C(p_n(q_{t,n}),q_{t,n};S) \leq \Delta_n(\frac{\theta}{\lambda})^{\frac{1}{\rho-1}},$$

where $q_{s,n}(q)$ is defined by $p_n(q)$ and Bayes' rule. It follows that, for any $\varepsilon > 0$, there exists an N_ε such that, for all $n > N_\varepsilon$,

$$P(||q_{t+\Delta_n,n}-q_{t,n}|| > \varepsilon) \leq K_\varepsilon \Delta_n,$$

for the constant $K_\varepsilon = 2m^{-1}\varepsilon^{-2}\theta^{\frac{1}{\rho-1}}$. By Theorem 3.21, Condition 1 in chapter 6 of Jacod and Shiryaev (2013), and the boundedness of $q_{t,n}$, it follows that the laws of $q_{t,n}$ are tight. By Prokhorov's theorem (Theorem 3.9 in chapter 6 of Jacod and Shiryaev (2013)), it follows that there exists a convergent sub-sequence. Pass to this sub-sequence, and let q_t denote the limiting stochastic process. By Proposition 1.1 in chapter 9 of Jacod and Shiryaev (2013), q_t is a martingale with respect to the filtration it generates. By Skorohod's representation theorem, there exists a probability space and random variables (which we will also denote with $q_{t,n}$ and q_t) such the convergence is almost sure. We refer to this probability space and these random variables in what follows.

Note that, by Bayes' rule, if $e_x^T q_{t,n} = 0$ for some $x \in X$ and time t, then $e_x^T q_{s,n} = 0$ for all $s > t$. By Proposition 2.9 and Corollary 2.38 in chapter 2 of Jacod and Shiryaev (2013), we can write the "good" version of the martingale with characteristics

$$B = -\int_0^t (\int_{\mathbb{R}^{|X|}\setminus\{0\}} \psi_s(x)xdx)dA_s$$

$$C = \int_0^t \Sigma_s dA_s$$

$$\nu = dA_s\psi_s(x).$$

Because beliefs remain in the simplex, $\psi_s(x)$ has support only on x such that $q_s + x \in \mathscr{P}(X)$. Relatedly, $\iota^T \Sigma_s = 0$. By the property mentioned above, $q_s + x \ll q_s$, and Σ_s can be decomposed as $\Sigma_s = D(q_{s^-})\sigma_s \sigma_s^T D(q_{s^-})$.

By the convexity of the cost function and Corollary 3,

$$C(p_n(q_{t,n}), q_{t,n}; S) \geq \sum_{s \in S} (e_s^T p_n(q_{t,n}) q_{t,n}) D^*(q_{s,n}(q_{t,n}) || q_{t,n}).$$

Defining the process, for arbitrary stopping time T,

$$D_{s,n} = \lim_{\varepsilon \to 0^+} D^*(q_{s^- + \varepsilon, n} || q_{s^-, n}),$$

$$D_{t,T,n} = E_t[\int_t^T D_{s,n} ds] \leq \theta^{\frac{1}{\rho-1}} \Delta_n E_t[\lceil \Delta_n^{-1}(T-t) \rceil],$$

we have by Ito's lemma, almost sure convergence, and the dominated convergence theorem,

$$D_{t,T} = \lim_{n \to \infty} D_{t,T,n} = E_t[\int_t^T \{\frac{1}{2} tr[\sigma_s \sigma_s^T k(q_{s^-})] + \int_{\mathbb{R}^{|X|} \setminus \{0\}} \psi_s(x) D^*(q_{s^-} + x || q_{s^-}) dx\} dA_s].$$

Hence, for all such stopping times T,

$$E_t[\int_t^T \{\frac{1}{2} tr[\sigma_s \sigma_s^T k(q_{s^-})] + \int_{\mathbb{R}^{|X|} \setminus \{0\}} \psi_s(x) D^*(q_{s^-} + x || q_{s^-}) dx\} dA_s] \leq (\frac{\theta}{\lambda})^{\frac{1}{\rho-1}} E_t[T-t].$$

Note also by this argument that

$$\lim_{n \to \infty} E_0[\int_0^{\tau_n} \Delta_n^{1-\rho} C(p_n(q_{t,n}), q_{t,n}; S)^\rho dt]$$

$$\geq E_t[\int_0^\tau \{\frac{1}{2} tr[\sigma_s \sigma_s^T k(q_{s^-})] + \int_{\mathbb{R}^{|X|} \setminus \{0\}} \psi_s(x) D^*(q_{s^-} + x || q_{s^-}) dx\}^\rho (\frac{dA_s}{ds})^\rho ds].$$

A.16 Proof of Theorem 4

Let m index a sequence of Markov optimal policies, $p_m^*(q)$, and of stopping times τ_m^*. Let $q_{t,n}^*$ denote the associated process for beliefs. By the uniform boundedness and convexity of the family of value functions $W(q,\lambda;\Delta_m)$, a uniformly convergent sub-sequence exists. Rockafellar (1970) Theorem 10.9 demonstrates that a uniformly convergent sub-sequence exists on the relative interior of the simplex, and Rockafellar (1970) Theorem 10.3 demonstrates that there is a unique extension to a convex and continuous function on the boundary of the simplex.

Pass to this sub-sequence, which (for simplicity) we also index by m, and let $W(q,\lambda)$ denote its limit. By Lemmas 8 and 9, the sequence of optimal policies and stopping time satisfies the conditions of Lemma 10. It follows by that lemma that

$$\lim_{n\to\infty} E_0[\int_0^{\tau_n^*} \Delta_n^{1-\rho} C(p_n^*(q_{t,n}^*), q_{t,n}^*; S)^\rho dt]$$

$$\geq E_t[\int_0^{\tau} \{\frac{1}{2} tr[\sigma_s^* \sigma_s^{*T} k(q_{s-}^*)] + \int_{\mathbb{R}^{|X|}\setminus\{0\}} \psi_s^*(x) D^*(q_{s-} + x || q_{s-}) dx\}^\rho \}^\rho (\frac{dA_s^*}{ds})^\rho ds],$$

where q_s^* is the limiting stochastic process and $\sigma_s^*, \psi_s^*, dA_s^*$ are associated with the characteristics of the martingale q_s^*.

We also have, by weak convergence,

$$\lim_{n\to\infty} E_0[\hat{u}(q_{\tau_n^*,n}) - (\kappa - \lambda c^\rho)\tau_n^*)] = E_0[\hat{u}(q_{\tau^*}) - (\kappa - \lambda c^\rho)\tau^*)].$$

Recall also the bound, for any stopping time T measurable with respect filtration generated by q_s^* ,

$$E_t[\int_t^T \{\frac{1}{2} tr[\sigma_s^* \sigma_s^{*T} k(q_{s-})] + \int_{\mathbb{R}^{|X|}\setminus\{0\}} \psi_s^*(x) D^*(q_{s-}^* + x || q_{s-}^*) dx\} dA_s^*] \leq (\frac{\theta}{\lambda})^{\frac{1}{\rho-1}} E_t[T-t].$$

It follows that

$$W(q,\lambda) \leq W^{+}(q,\lambda)$$

for all $q \in \mathscr{P}(X)$, where

$$W^{+}(q_t,\lambda) = \sup_{\{\sigma_s,\psi_s,dA_s,\tau\}} E_t[\hat{u}(q_\tau) - (\kappa - \lambda c^{\rho})(\tau - t)] -$$
$$-\frac{\lambda}{\rho} E_t[\int_t^{\tau} \{\frac{1}{2} tr[\sigma_s \sigma_s^T k(q_s)] + \int_{\mathbb{R}^{|X|}\setminus\{0\}} \psi_s(x) D^{*}(q_{s^-} + x || q_{s^-}) dx\}^{\rho} (\frac{dA_s}{ds})^{\rho} ds],$$

subject to the constraints, for all stopping times T measurable with respect filtration generated by q_s^{*} ,

$$E_t[\int_t^{T} \{\frac{1}{2} tr[\sigma_s \sigma_s^T k(q_t)] + \int_{\mathbb{R}^{|X|}\setminus\{0\}} \psi_s(x) D^{*}(q_{s^-} + x || q_{s^-}) dx\} dA_s] \leq (\frac{\theta}{\lambda})^{\frac{1}{\rho-1}} E_t[T - t]$$

and

$$E_0[\tau] \leq \bar{\tau},$$

and the evolution of beliefs as implied by the characteristics derived from σ_s, ψ_s, dA_s. Observe, by the arguments in the proof of Lemma 8, that $W^{+}(q,\lambda)$ is convex in q.

Also note that, for W^{+}, it is without loss of generality to set $dA_s = ds$. Scaling dA_s up and scaling $\sigma_s \sigma_s^T$ and ψ_s down, or vice versa, does not change the constraint, and setting $dA_s = 0$ is clearly sub-optimal by the assumption that $\kappa - \lambda c^{\rho} > 0$. Note also that there is a version of the optimal policies which satisfy the constraint everywhere:

$$\frac{1}{2} tr[\sigma_s \sigma_s^T k(q_{s^-})] + \int_{\mathbb{R}^{|X|}\setminus\{0\}} \psi_s(x) D^{*}(q_{s^-} + x || q_{s^-}) dx \leq (\frac{\theta}{\lambda})^{\frac{1}{\rho-1}}.$$

The associated Bellman equation, in the continuation region, is

$$0=\max_{\sigma_s,\psi_s} E[dW^+(q_s,\lambda)]-(\kappa-\lambda c^\rho)ds-\frac{\lambda}{\rho}\{\frac{1}{2}tr[\sigma_s\sigma_s^T k(q_s)]+\int_{\mathbb{R}^{|X|}\setminus\{0\}}\psi_s(x)D^*(q_{s^-}+x||q_{s^-})dx\}^\rho.$$

Let σ_s^+ and ψ_s^+ denote optimal policies for this problem (which we have yet to show are equal to σ_s^* and ψ_s^*). Suppose that the constraint does not bind, and consider a perturbation which scales $\sigma_s^+\sigma_s^{+T}$ and ψ_s^+ be some constant $(1+\varepsilon)$. Note that such a perturbation would also scale $E[dW^+]$ by $(1+\varepsilon)$, and that at least one of σ_s^+ and ψ_s^+ must be non-zero by the assumption that $\kappa-\lambda c^\rho>0$. The first order condition for this perturbation is

$$(\kappa-\lambda c^\rho)+\frac{\lambda}{\rho}\{\frac{1}{2}tr[\sigma_s^+\sigma_s^{+T}k(q_{s^-})]+\int_{\mathbb{R}^{|X|}\setminus\{0\}}\psi_s^+(x)D^*(q_{s^-}+x||q_{s^-})dx\}^\rho=$$
$$\lambda\{\frac{1}{2}tr[\sigma_s^+\sigma_s^{+T}k(q_{s^-})]+\int_{\mathbb{R}^{|X|}\setminus\{0\}}\psi_s^+(x)D^*(q_{s^-}+x||q_{s^-})dx\}^\rho,$$

which must hold at the optimal policies for this problem. It follows by the definition of θ (see the proof of Lemma 9) that the constraint binds.

Consider a sub-optimal policy which sets $\psi_s(x)=0$ and satisfies the constraint. The above FOC applies, and therefore we must have

$$tr[\tilde{\sigma}_s\tilde{\sigma}_s^T(D(q_{s^-})W_{qq}^+(q_{s^-},\lambda)D(q_{s^-})-\theta k(q_{s^-}))]\leq 0,$$

where W_{qq}^+ is understood in a distributional sense. It follows that, for all feasible x,

$$W^+(q_{s^-}+x,\lambda)-W^+(q_{s^-},\lambda)-x^TW_q^+(q_{s^-},\lambda;-x)\leq\frac{1}{2}\int_0^1 x^T\bar{k}(q_{s^-}+lx)xdl.$$

By Condition 6, this implies that

$$W^+(q_{s^-}+x,\lambda)-W^+(q_{s^-},\lambda)-x^TW_q^+(q_{s^-},\lambda;-x)\leq\theta D^*(q_{s^-}+x||q_{s^-}).$$

Hence, it is without loss of generality to assume that $\psi_s^+(x) = 0$ for all x. Note that, if there is a strict preference for gradual learning, the above inequality is strict for all non-zero x. As a result, in this case $\psi_s^+(x) = 0$ for all x. Note also that our control problem involves direct control of the diffusion coefficients, and hence satisfies the standard requirements for the existence and uniqueness of a strong solution to the resulting SDE (Pham (2009) sections 1.3 and 3.2).

Noting that $W^+(q,\lambda) \geq W(q,\lambda)$, it follows that if there exists a sequence of policies that converge to the stochastic process q_t^+, characterized by σ^+, and whose costs $\Delta_n^{-1}C(\cdot)$ converge to $\theta^{\frac{1}{\rho-1}}$, then such a sequence of policies achieves, in the limit, at least as much utility as any other sequence of policies. It would then be the case that there must be sequence of optimal policies that converges a.s. (as in Lemma 10) to some optimal policy of W^+ (not necessarily σ^c and ψ^c, but this does not matter for our argument). Note, however, that if there is a strict preference for gradual learning, and W^+ is achievable, all optimal policies of W^+ generate diffusions, and hence all convergent sub-sequences of beliefs induced by optimal policies in the discrete-time model must converge to diffusions.

Define the function

$$\Sigma^+(q) = D(q)\sigma^+(q)\sigma^+(q)^T D(q).$$

We will construct a sequence that converges to this diffusion process.

Consider the eigenvector decomposition of the matrix

$$L(q)\Upsilon(q)L(q)^T = \alpha_n(q)\Sigma^+(q),$$

where $\alpha_n(q) > 0$ is a scalar function of q. For each pair $(s_i, s_{i+1}) \in S$, where $i \in \{1,2,\ldots,|X|\}$

is an even integer, set $e_{s_i}^T r_n = e_{s_{i+1}}^T r_n = \frac{1}{2|X|}$, and set

$$q_{s_i,n}(q) - q =$$

$$q - q_{s_{i+1},n}(q) =$$

$$L(q)\Upsilon^{\frac{1}{2}}(q)e_i.$$

Set all other $e_s^T r_n = 0$. By construction,

$$\sum_{s \in S} (e_s^T r_{s,n})(q_{s,n}(q) - q) = 0,$$

and

$$\sum_{s \in S} (e_s^T r_n)(q_{s,n}(q) - q)(q_{s,n}(q) - q)^T = \alpha_n(q)\Sigma^+(q)$$

and

$$\sum_{s \in S} (e_s^T r_n) = 1.$$

We would like to have, for this policy, $C(p_n(q), q; S) = \Delta_n \theta^{\frac{1}{\rho-1}}$ always. Note that under this policy, $C(\cdot)$ is a function of α_n and q. By the convexity of $C(\cdot)$ and the definition of its derivatives,

$$C(\cdot) \geq \alpha_n(q)\frac{\partial C}{\partial \alpha}|_{\alpha=0} = \alpha_n(q)(\frac{1}{2}tr[k(q)\sigma^+(q)(\sigma^+(q))^T]),$$

and hence

$$C(\cdot) \geq \alpha_n(q)\theta^{\frac{1}{\rho-1}}.$$

It follows that $\alpha_n(q) \leq \Delta_n$, it is feasible to have $C(p_n(q), q; S) = \Delta_n \theta^{\frac{1}{\rho-1}}$.

Note, by the finiteness of $\Sigma^+(q)$ (due the positive definiteness of $\bar{k}(q)$), that $q_{s,n}(q) - q =$

$O(\Delta_n^{\frac{1}{2}})$. It follows from lemmas 11.2.1 and 11.2.2 in Stroock and Varadhan (2007) that the law of q_n under this process converges to a solution to the martingale problem associated with the coefficients $\sigma^+(q)$. By the uniqueness of this solution established earlier, this law is the law of q_t^+, a diffusion.

By the arguments in Amin and Khanna (1994), it is possible to construct from these sequences a Brownian motion and a probability space such that the random variable τ is a stopping time that is measurable with respect to the limiting stochastic process. It follows that $W(q,\lambda) = W^+(q,\lambda)$. Note that we have constructed a sequence of policies that converge to an optimal policy of $W(q,\lambda)$.

We next demonstrate equality of the primal and dual. We have shown that

$$W(q,\lambda) = E_0[\hat{u}(q_{\tau^*}) - (\kappa - \lambda c^\rho)\tau^*)] - \frac{\lambda}{\rho} E_0[\int_0^{\tau^*} (\frac{\theta}{\lambda})^{\frac{\rho}{\rho-1}} ds].$$

Recall the definition of θ,

$$\theta = \lambda(\rho \frac{\kappa - \lambda c^\rho}{\lambda(\rho-1)})^{\frac{\rho-1}{\rho}}.$$

Define λ^* by

$$\frac{\kappa - \lambda^* c^\rho}{\lambda^*(\rho-1)} = c^\rho,$$

which is

$$\lambda^* = \frac{\kappa}{\rho c^\rho}.$$

Note that $\lambda^* \in (0, \kappa c^{-\rho})$, as required. For this value of λ,

$$W(q_0,\lambda^*) = E_0[\hat{u}(q_{\tau^*}) - \kappa\tau^*],$$

and the limit of the constraint is satisfied:

$$\frac{\lambda^*}{\rho}E_0[\int_0^{\tau^*}(\frac{\theta}{\lambda^*})^{\frac{\rho}{\rho-1}}ds] = \lambda^* E_0[\int_0^{\tau^*} cds].$$

Consider a convergent sub-sequence of $V(q_0;\Delta_n)$ (which exists by the uniform boundedness and convexity of the problem), and denote its limit $V(q_0)$ (again, we will index this sequence by n). By the standard duality inequalities, for all λ,

$$V(q_0;\Delta_n) \leq W(q_0,\lambda;\Delta_n),$$

for all n, and therefore

$$V(q_0) \leq W(q_0,\lambda).$$

Consider the value function $\tilde{V}(q_0)$, which is the value function under the feasible optimal policies for $W(q_0,\lambda^*)$. It follows that $\tilde{V}(q_0) = W(q_0,\lambda^*)$, and $\tilde{V}(q_0) \leq V(q_0)$, and therefore $V(q_0) = W(q_0,\lambda^*)$.

We can define

$$\begin{aligned}\theta^* &= \lambda^*(\rho\frac{\kappa-\lambda^* c^\rho}{\lambda^*(\rho-1)})^{\frac{\rho-1}{\rho}} \\ &= \lambda^*\rho^{\frac{\rho-1}{\rho}}c^{\rho-1} \\ &= \frac{\kappa}{c}\rho^{-\rho^{-1}}.\end{aligned}$$

Note that every convergent sub-sequence of $V(q_0;\Delta_n)$ converges to the same function. By the boundedness of value function, it follows that

$$\begin{aligned} V(q_0) &= \lim_{\Delta \to 0^+} V(q_0;\Delta). \\ &= E_0[\hat{u}(q_{\tau^*}) - \kappa\tau^*]. \end{aligned}$$

The constraint can be written as

$$\frac{1}{2}tr[\sigma_s\sigma_s^T k(q_{s^-})] \leq (\frac{\theta^*}{\lambda^*})^{\frac{1}{\rho-1}},$$

with

$$(\frac{\theta^*}{\lambda^*})^{\frac{1}{\rho-1}} = (\rho^{1-\rho^{-1}} c^{\rho-1})^{\frac{1}{\rho-1}} = c\rho^{\rho^{-1}} = \chi.$$

The optimal policy satisfies this constraint, and hence it follows that the value function is the maximized over all policies satisfying

$$\frac{1}{2}tr[\sigma_s\sigma_s^T k(q_s)] \leq \chi,$$

concluding the proof.

B Appendix References

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