A Proofs

Proof of Proposition 1. The choice variable a'_{tD} does not appear in the Planner's objective function, so $a'_{tD} = 0$ at an optimum. Also, (5) must bind for every t at an optimum, so the planner's problem is equivalent to

$$\max_{\{\tilde{a}_{tD},\tilde{a}_{tI},\underline{a}_{tib(i)},a'_{tI}\}_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \Bigg[\delta \int_{[\varepsilon_L,\varepsilon_H]} \varepsilon a'_{tI}(d\varepsilon) + (1-\alpha-\delta)\bar{\varepsilon} a_{tI}$$

$$+ \int_{\mathcal{B}_t} \int\int \mathbb{I}_{\{i \leq b(i)\}} \left[\varepsilon_i \underline{a}_{tib(i)}\left(\varepsilon_i, \varepsilon_{b(i)}\right) + \varepsilon_{b(i)} \underline{a}_{tb(i)i}\left(\varepsilon_{b(i)}, \varepsilon_i\right)\right] dG(\varepsilon_i)\, dG\left(\varepsilon_{b(i)}\right) di\Bigg] y_t$$

$$\text{s.t. (2), (3), (6), (7) and } \delta \int_{[\varepsilon_L,\varepsilon_H]} a'_{tI}(d\varepsilon) \leq v a_{tD} + \delta a_{tI}.$$

Let W^* denote the maximum value of this problem. Then clearly, $W^* \leq \bar{W}^*$, where

$$\bar{W}^* = \max_{\{\tilde{a}_{tD},\tilde{a}_{tI}\}_{t=0}^{\infty}} \mathbb{E}_0 \Bigg[\sum_{t=0}^{\infty} \beta^t \Bigg[\int_{\mathcal{B}_t} \int\int \mathbb{I}_{\{i \leq b(i)\}} \max\left(\varepsilon, \varepsilon'\right) 2\tilde{a}_{tI} dG(\varepsilon)\, dG\left(\varepsilon'\right) di$$

$$+ \varepsilon_H \left(v\tilde{a}_{tD} + \delta \tilde{a}_{tI}\right) + (1-\alpha-\delta)\bar{\varepsilon}\tilde{a}_{tI}\Bigg]\pi y_t\Bigg] + w,$$

s.t. (3), where $w \equiv [\alpha\varepsilon_B + \delta\varepsilon_H + (1-\alpha-\delta)\bar{\varepsilon}](1-\pi) A^s \left(\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t y_t\right)$ and

$$\varepsilon_B \equiv \int\int \max\left(\varepsilon, \varepsilon'\right) dG(\varepsilon)\, dG\left(\varepsilon'\right).$$

Rearrange the expression for $\bar{W}^*$ and substitute (3) (at equality) to obtain

$$\bar{W}^* = \max_{\{\tilde{a}_{tI}\}_{t=0}^{\infty}} \mathbb{E}_0 \left\{\sum_{t=0}^{\infty} \beta^t \left\{\varepsilon_H A^s + [\alpha\varepsilon_B + \delta\varepsilon_H + (1-\alpha-\delta)\bar{\varepsilon} - \varepsilon_H]\tilde{a}_{tI}\right\}\pi y_t\right\} + w$$

$$= \{\pi\varepsilon_H + (1-\pi)[\alpha\varepsilon_B + \delta\varepsilon_H + (1-\alpha-\delta)\bar{\varepsilon}]\} A^s \left(\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t y_t\right).$$

The allocation $\tilde{a}_{tD} = A^s/v$, $\tilde{a}_{tI} = 0$, and $\underline{a}_{tib(i)}\left(\varepsilon_i, \varepsilon_{b(i)}\right) = \mathbb{I}_{\{\varepsilon_{b(i)}<\varepsilon_i\}} 2a_{tI} + \mathbb{I}_{\{\varepsilon_{b(i)}=\varepsilon_i\}} a^o$, where $a^o \in [0, 2a_{tI}]$, together with the Dirac measure defined in the statement of the proposition, achieve $\bar{W}^*$ and therefore solve the Planner's problem. ■

Proof of Lemma 1. Notice that (8) can be written as

$$W_t^D(\boldsymbol{a}_t) = \boldsymbol{\phi}_t \boldsymbol{a}_t + W_t^D(\mathbf{0}) \tag{59}$$

with $W_t^D(\mathbf{0})$ given by (14). With (59), (9) is equivalent to

$$\hat{W}_t^D(\boldsymbol{a}_t) = \max_{\hat{a}_t^m, \hat{a}_t^s} [\phi_t^m \hat{a}_t^m + \phi_t^s \hat{a}_t^s + \xi(a_t^m + p_t a_t^s - \hat{a}_t^m - p_t \hat{a}_t^s) + \varsigma_m \hat{a}_t^m + \varsigma_s \hat{a}_t^s] + W_t^D(\mathbf{0})$$

where ξ is a Lagrange multiplier on the budget constraint $\hat{a}_t^m + p_t \hat{a}_t^s \leq a_t^m + p_t a_t^s$, and ς_m and ς_s are the multipliers on the nonnegativity constraints $\hat{a}_t^m \geq 0$ and $\hat{a}_t^s \geq 0$. The corresponding first-order necessary and sufficient conditions for $\hat{a}_t^m$ and $\hat{a}_t^s$ are

$$-\xi + \phi_t^m + \varsigma_m = 0 \tag{60}$$

$$-\xi p_t + \phi_t^s + \varsigma_s = 0 \tag{61}$$

$$\xi(a_t^m + p_t a_t^s - \hat{a}_t^m - p_t \hat{a}_t^s) = 0. \tag{62}$$

Clearly $\hat{a}_t^m = \hat{a}_t^s = 0$ is the solution if and only if $a_t^m = a_t^s = 0$, but more generally the solution could take one of three forms: (i) $\varsigma_s = 0 < \varsigma_m$, (ii) $\varsigma_s = \varsigma_m = 0$, or (iii) $\varsigma_m = 0 < \varsigma_s$. In case (i), (60)-(62) imply $\hat{a}_t^m = 0$, $\hat{a}_t^s = a_t^s + \frac{1}{p_t} a_t^m$, and $p_t \phi_t^m < \phi_t^s$. In case (ii), (60)-(62) imply $\hat{a}_t^m \in [0, a_t^m + p_t a_t^s]$, $\hat{a}_t^s = a_t^s + \frac{1}{p_t}(a_t^m - \hat{a}_t^m)$, and $\phi_t^s = p_t \phi_t^m$. In case (iii), (60)-(62) imply $\hat{a}_t^s = 0$, $\hat{a}_t^m = a_t^m + p_t a_t^s$, and $\phi_t^s < p_t \phi_t^m$. The expressions for $\hat{a}_{td}^m$ and $\hat{a}_{td}^s$ in Lemma 1 follow from these three cases. The value function (13) is obtained by substituting the optimal portfolio $(\hat{a}_{td}^m, \hat{a}_{td}^s)$ into (9). ■

Proof of Lemma 2. (i) Notice that (11) can be written as

$$W_t^I(\boldsymbol{a}_t) = \boldsymbol{\phi}_t \boldsymbol{a}_t + W_t^I(\mathbf{0}) \tag{63}$$

where

$$W_t^I(\mathbf{0}) = T_t + \max_{\tilde{\boldsymbol{a}}_{t+1} \in \mathbb{R}_+^2} \left[\beta \mathbb{E}_t \int V_{t+1}^I(\boldsymbol{a}_{t+1}, \varepsilon)\, dG(\varepsilon) - \boldsymbol{\phi}_t \tilde{\boldsymbol{a}}_{t+1} \right]$$
$$\text{s.t. } \boldsymbol{a}_{t+1} = (\tilde{a}_{t+1}^m, \pi \tilde{a}_{t+1}^s + (1-\pi) A^s).$$

With (13) and (63) the problem of the investor when he makes the ultimatum offer becomes

$$\max_{\overline{a}_{ti^*}^m, \overline{a}_{ti^*}^s, \overline{a}_{td}^m, \overline{a}_{td}^s} [\varepsilon y_t \overline{a}_{ti^*}^s + \phi_t^m \overline{a}_{ti^*}^m + \phi_t^s \overline{a}_{ti^*}^s]$$

$$\text{s.t. } \overline{a}_{ti^*}^m + \overline{a}_{td}^m + p_t(\overline{a}_{ti^*}^s + \overline{a}_{td}^s) \leq a_{ti}^m + a_{td}^m + p_t(a_{ti}^s + a_{td}^s)$$
$$\overline{a}_{td}^m + p_t \overline{a}_{td}^s \geq a_{td}^m + p_t a_{td}^s$$
$$\overline{a}_{ti^*}^m, \overline{a}_{ti^*}^s, \overline{a}_{td}^m, \overline{a}_{td}^s \in \mathbb{R}_+.$$

The corresponding Lagrangian is

$$\mathcal{L} = (\phi_t^m + \varsigma_i^m - \xi)\,\overline{a}_{ti^*}^m + (\varepsilon y_t + \phi_t^s + \varsigma_i^s - \xi p_t)\,\overline{a}_{ti^*}^s$$
$$+ (\rho + \varsigma_d^m - \xi)\,\overline{a}_{td}^m + (\rho p_t + \varsigma_d^s - \xi p_t)\,\overline{a}_{td}^s + K,$$

where $K \equiv \xi\,[a_{ti}^m + a_{td}^m + p_t(a_{ti}^s + a_{td}^s)] - \rho\,(a_{td}^m + p_t a_{td}^s)$, $\xi \in \mathbb{R}_+$ is the Lagrange multiplier associated with the budget constraint, $\rho \in \mathbb{R}_+$ is the multiplier on the dealer's individual rationality constraint, and $\varsigma_i^m, \varsigma_i^s, \varsigma_d^m, \varsigma_d^s \in \mathbb{R}_+$ are the multipliers for the nonnegativity constraints on $\overline{a}_{ti^*}^m$, $\overline{a}_{ti^*}^s$, $\overline{a}_{td}^m$, $\overline{a}_{td}^s$, respectively. The first-order necessary and sufficient conditions are

$$\phi_t^m + \varsigma_i^m - \xi = 0 \tag{64}$$
$$\varepsilon y_t + \phi_t^s + \varsigma_i^s - \xi p_t = 0 \tag{65}$$
$$\rho + \varsigma_d^m - \xi = 0 \tag{66}$$
$$\rho p_t + \varsigma_d^s - \xi p_t = 0 \tag{67}$$

and the complementary slackness conditions

$$\xi\,\{a_{ti}^m + a_{td}^m + p_t(a_{ti}^s + a_{td}^s) - [\overline{a}_{ti^*}^m + \overline{a}_{td}^m + p_t(\overline{a}_{ti^*}^s + \overline{a}_{td}^s)]\} = 0 \tag{68}$$
$$\rho\,[\overline{a}_{td}^m + p_t\overline{a}_{td}^s - (a_{td}^m + p_t a_{td}^s)] = 0 \tag{69}$$
$$\varsigma_i^m\overline{a}_{ti^*}^m = 0 \tag{70}$$
$$\varsigma_i^s\overline{a}_{ti^*}^s = 0 \tag{71}$$
$$\varsigma_d^m\overline{a}_{td}^m = 0 \tag{72}$$
$$\varsigma_d^s\overline{a}_{td}^s = 0. \tag{73}$$

First, notice that $\xi > 0$ at an optimum. To see this, assume the contrary, i.e., $\xi = 0$. Then (65) implies $\varepsilon y_t + \phi_t^s = -\varsigma_i^s \leq 0$ which is a contradiction since $\varepsilon y_t + \phi_t^s > 0$. If $\rho > 0$, then (69) implies

$$\overline{a}_{td}^m + p_t\overline{a}_{td}^s = a_{td}^m + p_t a_{td}^s. \tag{74}$$

If instead $\rho = 0$, then (66) and (67) imply $\varsigma_d^m = \xi > 0$ and $\varsigma_d^s = \xi p_t > 0$, which (using (72) and (73)) in turn imply $\overline{a}_{td}^m = \overline{a}_{td}^s = 0$. This can only be a solution if $a_{td}^m + p_t a_{td}^s = 0$ (since $\overline{a}_{td}^m + p_t\overline{a}_{td}^s \geq a_{td}^m + p_t a_{td}^s$ must hold at an optimum) in which case (74) also holds. Thus, we conclude that (74) must always hold at an optimum (and with $\rho > 0$ unless $a_{td}^m + p_t a_{td}^s = 0$). Since $\xi > 0$, (68) and (74) imply

$$\overline{a}_{ti^*}^m + p_t\overline{a}_{ti^*}^s = a_{ti}^m + p_t a_{ti}^s. \tag{75}$$

From (74) it is immediate that if $a^m_{td}+p_t a^s_{td}=0$, then $\overline{a}^m_{td}=\overline{a}^s_{td}=0$. So suppose $a^m_{td}+p_t a^s_{td}>0$. In this case ς^m_d and ς^s_d cannot both be strictly positive. (To see this, assume the contrary, i.e., that $\varsigma^m_d>0$ and $\varsigma^s_d>0$. Then (72) and (73) imply $\overline{a}^m_{td}=\overline{a}^s_{td}=0$, and (74) implies $a^m_{td}+p_t a^s_{td}=0$, a contradiction.) Moreover, conditions (66) and (67) imply $\varsigma^s_d=\varsigma^m_d p_t$, so $\varsigma^s_d=\varsigma^m_d=0$ must hold at an optimum. Hence when making the ultimatum offer, the investor is indifferent between offering the dealer any nonnegative pair $(\overline{a}^m_{td},\overline{a}^s_{td})$ that satisfies (74).

From (75) it is immediate that $\overline{a}^m_{ti^*}=\overline{a}^s_{ti^*}=0$ if $a^m_{ti}+p_t a^s_{ti}=0$. So suppose $a^m_{ti}+p_t a^s_{ti}>0$. In this case ς^m_i and ς^s_i cannot both be strictly positive (if they were, then (70) and (71) would imply $\overline{a}^m_{ti^*}=\overline{a}^s_{ti^*}=0$, and in turn (75) would imply $a^m_{ti}+p_t a^s_{ti}=0$, a contradiction). There are three possible cases: (a) $\varsigma^s_i=0<\varsigma^m_i$, (b) $\varsigma^s_i=\varsigma^m_i=0$, or (c) $\varsigma^m_i=0<\varsigma^s_i$. In every case, (64) and (65) imply

$$\varepsilon y_t+\phi^s_t+\varsigma^s_i=p_t\phi^m_t+p_t\varsigma^m_i. \tag{76}$$

In case (a), (70) implies $\overline{a}^m_{ti^*}=0$, (75) implies $\overline{a}^s_{ti^*}=a^m_{ti}/p_t+a^s_{ti}$, and (76) implies that ε must satisfy $\varepsilon>\varepsilon^*_t$, where ε^*_t is as defined in (15). In case (b), (76) implies that ε must satisfy $\varepsilon=\varepsilon^*_t$ and the investor is indifferent between making any offer that leaves him with a nonnegative post-trade portfolio $(\overline{a}^m_{ti^*},\overline{a}^s_{ti^*})$ that satisfies (75). In case (c), (71) implies $\overline{a}^s_{ti^*}=0$, (75) implies $\overline{a}^m_{ti^*}=a^m_{ti}+p_t a^s_{ti}$, and (76) implies that ε must satisfy $\varepsilon<\varepsilon^*_t$. The first, second, and third lines on the right side of the expressions for $\overline{a}^m_{ti^*}$, $\overline{a}^s_{ti^*}$, $\overline{a}^m_{td}$, and $\overline{a}^s_{td}$ in part (i) of the statement of the lemma correspond cases (a), (b), and (c), respectively.

(ii) With (13) and (63) the problem of the dealer when it is his turn to make the ultimatum offer is equivalent to

$$\max_{\overline{a}^m_{ti},\overline{a}^s_{ti},\overline{a}^m_{td^*},\overline{a}^s_{td^*}} \bar{\phi}_t\left[\overline{a}^m_{td^*}+p_t\overline{a}^s_{td^*}\right]$$

$$\text{s.t. } \overline{a}^m_{ti}+\overline{a}^m_{td^*}+p_t(\overline{a}^s_{ti}+\overline{a}^s_{td^*})\le a^m_{ti}+a^m_{td}+p_t(a^s_{ti}+a^s_{td}) \tag{77}$$

$$\phi^m_t\overline{a}^m_{ti}+(\varepsilon y_t+\phi^s_t)\,\overline{a}^s_{ti}\ge\phi^m_t a^m_{ti}+(\varepsilon y_t+\phi^s_t)\,a^s_{ti} \tag{78}$$

$$\overline{a}^m_{ti},\overline{a}^s_{ti},\overline{a}^m_{td^*},\overline{a}^s_{td^*}\in\mathbb{R}_+.$$

The corresponding Lagrangian is

$$\begin{aligned}\mathcal{L}'=&(\bar{\phi}_t+\varsigma^m_d-\xi)\overline{a}^m_{td^*}+(\bar{\phi}_t p_t+\varsigma^s_d-\xi p_t)\overline{a}^s_{td^*}\\&+(\rho\phi^m_t+\varsigma^m_i-\xi)\,\overline{a}^m_{ti}+\left[\rho\left(\varepsilon y_t+\phi^s_t\right)+\varsigma^s_i-\xi p_t\right]\overline{a}^s_{ti}+K',\end{aligned}$$

where $K'\equiv\xi\left[a^m_{ti}+a^m_{td}+p_t\left(a^s_{ti}+a^s_{td}\right)\right]-\rho\left[\phi^m_t a^m_{ti}+\left(\varepsilon y_t+\phi^s_t\right)a^s_{ti}\right]$, $\xi\in\mathbb{R}_+$ is the Lagrange multiplier associated with the budget constraint, $\rho\in\mathbb{R}_+$ is the multiplier on the investor's individual rationality constraint, and $\varsigma_i^m, \varsigma_i^s, \varsigma_d^m, \varsigma_d^s \in \mathbb{R}_+$ are the multipliers for the nonnegativity constraints on $\overline{a}_{ti}^m$, $\overline{a}_{ti}^s$, $\overline{a}_{td^*}^m$, $\overline{a}_{td^*}^s$, respectively. The first-order necessary and sufficient conditions are

$$\bar{\phi}_t + \varsigma_d^m - \xi = 0 \tag{79}$$

$$\bar{\phi}_t p_t + \varsigma_d^s - \xi p_t = 0 \tag{80}$$

$$\rho \phi_t^m + \varsigma_i^m - \xi = 0 \tag{81}$$

$$\rho \left(\varepsilon y_t + \phi_t^s\right) + \varsigma_i^s - \xi p_t = 0 \tag{82}$$

and the complementary slackness conditions

$$\xi \left\{a_{ti}^m + a_{td}^m + p_t(a_{ti}^s + a_{td}^s) - [\overline{a}_{ti}^m + \overline{a}_{td^*}^m + p_t(\overline{a}_{ti}^s + \overline{a}_{td^*}^s)]\right\} = 0 \tag{83}$$

$$\rho \left\{\phi_t^m \overline{a}_{ti}^m + \left(\varepsilon y + \phi_t^s\right) \overline{a}_{ti}^s - [\phi_t^m a_{ti}^m + \left(\varepsilon y_t + \phi_t^s\right) a_{ti}^s]\right\} = 0 \tag{84}$$

$$\varsigma_i^m \overline{a}_{ti}^m = 0 \tag{85}$$

$$\varsigma_i^s \overline{a}_{ti}^s = 0 \tag{86}$$

$$\varsigma_d^m \overline{a}_{td^*}^m = 0 \tag{87}$$

$$\varsigma_d^s \overline{a}_{td^*}^s = 0. \tag{88}$$

First, notice that $\xi > 0$ at an optimum. To see this, note that if $\xi = 0$ then (79) implies $\bar{\phi}_t + \varsigma_d^m = 0$ which is a contradiction since the left side is strictly positive ($\bar{\phi}_t > 0$ and $\varsigma_d^m \geq 0$ in a monetary equilibrium). Hence, at an optimum,

$$\overline{a}_{ti}^m + \overline{a}_{td^*}^m + p_t(\overline{a}_{ti}^s + \overline{a}_{td^*}^s) = a_{ti}^m + a_{td}^m + p_t(a_{ti}^s + a_{td}^s). \tag{89}$$

Second, observe that conditions (79) and (80), imply $p_t \varsigma_d^m = \varsigma_d^s$, so ς_d^m and ς_d^s have the same sign, i.e., either both are positive or both are zero.

If $\rho = 0$, then (81) and (82) imply $\varsigma_i^m = \xi > 0$ and $\varsigma_i^s = \xi p_t > 0$, which (using (85) and (86)) in turn imply $\overline{a}_{ti}^m = \overline{a}_{ti}^s = 0$. From the buyer's individual rationality constraint (78) it follows that this can be a solution only if $\phi_t^m a_{ti}^m + \left(\varepsilon y_t + \phi_t^s\right) a_{ti}^s = 0$, or equivalently only if $a_{ti}^m = a_{ti}^s = 0$. To obtain $(\overline{a}_{td^*}^m, \overline{a}_{td^*}^s)$, consider two cases: (a) $\varsigma_d^m = \varsigma_d^s = 0$, in which case $(\overline{a}_{td^*}^m, \overline{a}_{td^*}^s)$ need only satisfy $\overline{a}_{td^*}^m + p_t \overline{a}_{td^*}^s = a_{td}^m + p_t a_{td}^s$, or (b) $\varsigma_d^m > 0$ and $\varsigma_d^s > 0$, in which case $\overline{a}_{td^*}^m = \overline{a}_{td^*}^s = 0$, which according to (77), is only possible if $a_{td}^m = a_{td}^s = 0$. It is easy to see that the solution for case (a) can be obtained from the expressions for $\overline{a}_{ti}^m$, $\overline{a}_{ti}^s$, $\overline{a}_{td^*}^m$, and $\overline{a}_{td^*}^s$ in part (ii) of the statement of the lemma simply by setting $a^m_{ti} = a^s_{ti} = 0$, and the solution for case (b) can be obtained similarly, by setting $a^m_{ti} = a^s_{ti} = a^m_{td} = a^s_{td} = 0$.

If $\rho > 0$, then (84) implies

$$\phi^m_t \overline{a}^m_{ti} + (\varepsilon y_t + \phi^s_t) \overline{a}^s_{ti} = \phi^m_t a^m_{ti} + (\varepsilon y + \phi^s_t) a^s_{ti}. \tag{90}$$

There are eight possible configurations of to be considered: [Configuration 1] $\varsigma^s_i = \varsigma^m_d = \varsigma^s_d = 0 < \varsigma^m_i$. In this case (85) implies $\overline{a}^m_{ti} = 0$. Conditions (79)-(82) imply $\varsigma^m_i = (\varepsilon - \varepsilon^*_t) \bar{\phi}_t y_t / (\varepsilon y_t + \phi^s_t)$, and therefore $\varepsilon^*_t < \varepsilon$. Then from (89) and (90) it follows that

$$\overline{a}^s_{ti} = a^s_{ti} + \left(\frac{\varepsilon^*_t y_t + \phi^s_t}{\varepsilon y_t + \phi^s_t} \right) \frac{1}{p_t} a^m_{ti}$$

and $(\overline{a}^m_{td^*}, \overline{a}^s_{td^*})$ is any nonnegative pair that satisfies

$$\overline{a}^m_{td^*} + p_t \overline{a}^s_{td^*} = a^m_{td} + p_t a^s_{td} + \frac{(\varepsilon - \varepsilon^*_t) y_t}{\varepsilon y_t + \phi^s_t} a^m_{ti}.$$

[Configuration 2] $\varsigma^m_i = \varsigma^s_i = \varsigma^m_d = \varsigma^s_d = 0$. In this case conditions (79)-(82) imply $\varepsilon = \varepsilon^*_t$, and (89) and (90) yield

$$\overline{a}^m_{ti} + p_t \overline{a}^s_{ti} = a^m_{ti} + p_t a^s_{ti} \tag{91}$$

$$\overline{a}^m_{td^*} + p_t \overline{a}^s_{td^*} = a^m_{td} + p_t a^s_{td}. \tag{92}$$

Hence the dealer is indifferent between making any offer $(\overline{a}^m_{ti}, \overline{a}^s_{ti}, \overline{a}^m_{td^*}, \overline{a}^s_{td^*})$ such that $(\overline{a}^m_{ti}, \overline{a}^s_{ti}) \in \mathbb{R}_+$ satisfies (91), and $(\overline{a}^m_{td^*}, \overline{a}^s_{td^*}) \in \mathbb{R}_+$ satisfies (92). [Configuration 3] $\varsigma^m_i = \varsigma^m_d = \varsigma^s_d = 0 < \varsigma^s_i$. In this case condition (86) implies $\overline{a}^s_{ti} = 0$. Conditions (81) and (82) imply $\varsigma^s_i = (\varepsilon^*_t - \varepsilon) y_t \rho$, and therefore $\varepsilon < \varepsilon^*_t$. Then from (89) and (90) it follows that

$$\overline{a}^m_{ti} = a^m_{ti} + \frac{\varepsilon y_t + \phi^s_t}{\varepsilon^*_t y_t + \phi^s_t} p_t a^s_{ti}$$

and $(\overline{a}^m_{td^*}, \overline{a}^s_{td^*})$ is any nonnegative pair that satisfies

$$\overline{a}^m_{td^*} + p_t \overline{a}^s_{td^*} = a^m_{td} + p_t a^s_{td} + \frac{(\varepsilon^*_t - \varepsilon) y_t}{\varepsilon^*_t y_t + \phi^s_t} p_t a^s_{ti}.$$

[Configuration 4] $\varsigma^m_d = \varsigma^s_d = 0$, $0 < \varsigma^m_i$ and $0 < \varsigma^s_i$. In this case conditions (85) and (86) imply $\overline{a}^m_{ti} = \overline{a}^s_{ti} = 0$, which according to (90), is only possible if $a^m_{ti} = a^s_{ti} = 0$. Then $(\overline{a}^m_{td^*}, \overline{a}^s_{td^*})$ is any nonnegative pair that satisfies (92). [Configuration 5] $\varsigma^s_i = 0 < \varsigma^m_i$, $0 < \varsigma^m_d$ and $0 < \varsigma^s_d$. In this case conditions (85), (87) and (88) imply $\overline{a}_{ti}^{m} = \overline{a}_{td^*}^{m} = \overline{a}_{td^*}^{s} = 0$. Conditions (81) and (82) imply $\varepsilon_t^* < \varepsilon$. Then from (89) and (90) it follows that the following condition must hold:

$$a_{td}^{s} + \frac{1}{p_t} a_{td}^{m} = -\left[\frac{(\varepsilon - \varepsilon_t^*)\, y_t}{(\varepsilon - \varepsilon_t^*)\, y_t + p_t \phi_t^m}\right] \frac{1}{p_t} a_{ti}^{m}.$$

The term on the left side of the equality is nonnegative and the term on the right side of the equality is nonpositive (since $\varepsilon_t^* < \varepsilon$), so this condition can hold only if $a_{ti}^m = a_{td}^m = a_{td}^s = 0$. Therefore (89) implies $\overline{a}_{ti}^{s} = a_{ti}^{s}$. [Configuration 6] $\varsigma_i^m = \varsigma_i^s = 0$, $0 < \varsigma_d^m$ and $0 < \varsigma_d^s$. In this case conditions (87) and (88) imply $\overline{a}_{td^*}^{m} = \overline{a}_{td^*}^{s} = 0$. Conditions (81) and (82) imply $\varepsilon = \varepsilon_t^*$, and in turn conditions (89) and (90) imply $a_{td}^m + p_t a_{td}^s = 0$, or equivalently, $a_{td}^m = a_{td}^s = 0$ must hold, and $(\overline{a}_{ti}^{m}, \overline{a}_{ti}^{s})$ is any nonnegative pair that satisfies (91). [Configuration 7] $\varsigma_i^m = 0 < \varsigma_i^s$, $0 < \varsigma_d^s$ and $0 < \varsigma_d^s$. In this case conditions (86)-(88) imply $\overline{a}_{ti}^{s} = \overline{a}_{td^*}^{m} = \overline{a}_{td^*}^{s} = 0$. Conditions (81) and (82) imply $\varepsilon < \varepsilon_t^*$. Then from (89) and (90) it follows that the following condition must hold:

$$\phi_t^m \left(a_{td}^m + p_t a_{td}^s\right) = -\left(\varepsilon_t^* - \varepsilon\right) y_t a_{ti}^s.$$

The term on the left side of the equality is nonnegative and the term on the right side of the equality is nonpositive (since $\varepsilon < \varepsilon_t^*$), so this condition can hold only if $\phi_t^m \left(a_{td}^m + p_t a_{td}^s\right) = a_{ti}^s = 0$. Therefore (90) implies $\overline{a}_{ti}^{m} = a_{ti}^{m}$. [Configuration 8] $0 < \varsigma_i^m$, $0 < \varsigma_i^s$, $0 < \varsigma_d^s$ and $0 < \varsigma_d^s$. In this case conditions (85)-(88) imply $\overline{a}_{ti}^{m} = \overline{a}_{ti}^{s} = \overline{a}_{td^*}^{m} = \overline{a}_{td^*}^{s} = 0$, which according to (89) is only possible, and the only possible solution if $a_{ti}^m = a_{ti}^s = a_{td}^m = a_{td}^s = 0$. To conclude, notice that the solutions for Configurations 1, 2, and 3, correspond to the first, second, and third lines of the expressions for $\overline{a}_{ti}^{m}$, $\overline{a}_{ti}^{s}$, $\overline{a}_{td^*}^{m}$, and $\overline{a}_{td^*}^{s}$ in part (*ii*) of the statement of the lemma. Similarly, the solution for Configuration 5 corresponds to the first line of the expressions for $\overline{a}_{ti}^{m}$, $\overline{a}_{ti}^{s}$, $\overline{a}_{td^*}^{m}$, and $\overline{a}_{td^*}^{s}$ in part (*ii*) of the statement of the lemma, with $a_{ti}^m = a_{td}^m = a_{td}^s = 0$. The solution for Configuration 6 corresponds to the second line of the expressions for $\overline{a}_{ti}^{m}$, $\overline{a}_{ti}^{s}$, $\overline{a}_{td^*}^{m}$, and $\overline{a}_{td^*}^{s}$ in part (*ii*) of the statement of the lemma, with $a_{td}^m = a_{td}^s = 0$. The solution for Configuration 7 corresponds to the third line of the expressions for $\overline{a}_{ti}^{m}$, $\overline{a}_{ti}^{s}$, $\overline{a}_{td^*}^{m}$, and $\overline{a}_{td^*}^{s}$ in part (*ii*) of the statement of the lemma, with $\phi_t^m \left(a_{td}^m + p_t a_{td}^s\right) = a_{ti}^s = 0$. Finally, it is easy to see that the solution for Configuration 4 can be obtained from the expressions for $\overline{a}_{ti}^{m}$, $\overline{a}_{ti}^{s}$, $\overline{a}_{td^*}^{m}$, and $\overline{a}_{td^*}^{s}$ in part (*ii*) of the statement of the lemma simply by setting $a_{ti}^m = a_{ti}^s = 0$, and the solution for case Configuration 8 can be obtained similarly, by setting $a_{ti}^m = a_{ti}^s = a_{td}^m = a_{td}^s = 0$. ■

Proof of Lemma 3. With (63) investor i's problem when choosing his take-it-or-leave it offer to investor j reduces to

$$\max_{\underline{a}^m_{ti*},\underline{a}^s_{ti*},\underline{a}^m_{tj},\underline{a}^s_{tj}} \left[\left(\varepsilon_i y_t + \phi^s_t\right)\underline{a}^s_{ti*} + \phi^m_t \underline{a}^m_{ti*}\right]$$

$$\text{s.t. } \underline{a}^m_{ti*} + \underline{a}^m_{tj} \le a^m_{ti} + a^m_{tj}$$

$$\underline{a}^s_{ti*} + \underline{a}^s_{tj} \le a^s_{ti} + a^s_{tj}$$

$$\varepsilon_j y_t \underline{a}^s_{tj} + \phi^m_t \underline{a}^m_{tj} + \phi^s_t \underline{a}^s_{tj} \ge \varepsilon_j y_t a^s_{tj} + \phi^m_t a^m_{tj} + \phi^s_t a^s_{tj}$$

$$\underline{a}^m_{ti*}, \underline{a}^s_{ti*}, \underline{a}^m_{tj}, \underline{a}^s_{tj} \in \mathbb{R}_+.$$

If $\phi^m_t = 0$, then $\underline{a}^s_{ti*} = a^s_{ti}$ and $\underline{a}^s_{tj} = a^s_{tj}$ (the bargaining outcome is no trade between investors i and j) so suppose $\phi^m_t > 0$ for the rest of the proof. The Lagrangian corresponding to investor i's problem is

$$\mathcal{L} = \left(\phi^m_t + \varsigma^m_i - \xi^m\right)\underline{a}^m_{ti*} + \left(\varepsilon_i y_t + \phi^s_t + \varsigma^s_i - \xi^s\right)\underline{a}^s_{ti*}$$
$$+ \left(\rho\phi^m_t + \varsigma^m_j - \xi^m\right)\underline{a}^m_{tj} + \left[\rho\left(\varepsilon_j y_t + \phi^s_t\right) + \varsigma^s_j - \xi^s\right]\underline{a}^s_{tj} + K'',$$

where $K'' \equiv \xi^m(a^m_{ti} + a^m_{tj}) + \xi^s(a^s_{ti} + a^s_{tj}) - \rho(\varepsilon_j y_t a^s_{tj} + \phi^m_t a^m_{tj} + \phi^s_t a^s_{tj})$, $\xi^m \in \mathbb{R}_+$ is the multiplier associated with the bilateral constraint on money holdings, $\xi^s \in \mathbb{R}_+$ is the multiplier associated with the bilateral constraint on equity holdings, $\rho \in \mathbb{R}_+$ is the multiplier on investor j's individual rationality constraint, and $\varsigma^m_i, \varsigma^s_i, \varsigma^m_j, \varsigma^s_j \in \mathbb{R}_+$ are the multipliers for the nonnegativity constraints on $\overline{a}^m_{ti*}$, $\overline{a}^s_{ti*}$, $\overline{a}^m_{tj}$, $\overline{a}^s_{tj}$, respectively. The first-order necessary and sufficient conditions are

$$\phi^m_t + \varsigma^m_i - \xi^m = 0 \tag{93}$$

$$\varepsilon_i y_t + \phi^s_t + \varsigma^s_i - \xi^s = 0 \tag{94}$$

$$\rho\phi^m_t + \varsigma^m_j - \xi^m = 0 \tag{95}$$

$$\rho\left(\varepsilon_j y_t + \phi^s_t\right) + \varsigma^s_j - \xi^s = 0 \tag{96}$$

and the complementary slackness conditions

$$\xi^m(a_{ti}^m + a_{tj}^m - \underline{a}_{ti^*}^m - \underline{a}_{tj}^m) = 0 \tag{97}$$

$$\xi^s(a_{ti}^s + a_{tj}^s - \underline{a}_{ti^*}^s - \underline{a}_{tj}^s) = 0 \tag{98}$$

$$\rho(\varepsilon_j y_t \underline{a}_{tj}^s + \phi_t^m \underline{a}_{tj}^m + \phi_t^s \underline{a}_{tj}^s - \varepsilon_j y_t a_{tj}^s - \phi_t^m a_{tj}^m - \phi_t^s a_{tj}^s) = 0 \tag{99}$$

$$\varsigma_i^m \underline{a}_{ti^*}^m = 0 \tag{100}$$

$$\varsigma_i^s \underline{a}_{ti^*}^s = 0 \tag{101}$$

$$\varsigma_j^m \underline{a}_{tj}^m = 0 \tag{102}$$

$$\varsigma_j^s \underline{a}_{tj}^s = 0. \tag{103}$$

If $\xi^m = 0$, (93) implies $0 < \phi_t^m = -\varsigma_i^m \leq 0$, a contradiction. If $\xi^s = 0$, (94) implies $0 < \varepsilon_i y_t + \phi_t^s = -\varsigma_i^s \leq 0$, another contradiction. Hence $\xi^m > 0$ and $\xi^s > 0$, so (97) and (98) imply

$$\underline{a}_{ti^*}^m + \underline{a}_{tj}^m = a_{ti}^m + a_{tj}^m \tag{104}$$

$$\underline{a}_{ti^*}^s + \underline{a}_{tj}^s = a_{ti}^s + a_{tj}^s. \tag{105}$$

If $\rho = 0$, (95) and (96) imply $\varsigma_j^m = \xi^m > 0$ and $\varsigma_j^s = \xi^s > 0$, and (102) and (103) imply $\underline{a}_{tj}^m = \underline{a}_{tj}^s = 0$. From investor's j individual rationality constraint, this can only be a solution if $a_{tj}^m = a_{tj}^s = 0$, and if this is the case (97) and (98) imply $(\underline{a}_{ti^*}^m, \underline{a}_{ti^*}^s) = (a_{ti}^m, a_{ti}^s)$. Hereafter suppose $\rho > 0$ which using (99) implies

$$\phi_t^m \underline{a}_{tj}^m + (\varepsilon_j y_t + \phi_t^s)\underline{a}_{tj}^s = \phi_t^m a_{tj}^m + (\varepsilon_j y_t + \phi_t^s) a_{tj}^s. \tag{106}$$

If $\varsigma_i^m > 0$ and $\varsigma_j^m > 0$, (100) and (102) imply $\underline{a}_{ti^*}^m = \underline{a}_{tj}^m = 0$ which by (104), is only possible if $a_{ti}^m = a_{tj}^m = 0$. But then (106) implies $\underline{a}_{tj}^s = a_{tj}^s$, and (105) implies $\underline{a}_{ti^*}^s = a_{ti}^s$. Similarly, if $\varsigma_i^s > 0$ and $\varsigma_j^s > 0$, (101) and (103) imply $\underline{a}_{ti^*}^s = \underline{a}_{tj}^s = 0$ which by (105), is only possible if $a_{ti}^s = a_{tj}^s = 0$. But then (106) implies $\underline{a}_{tj}^m = a_{tj}^m$, and (104) implies $\underline{a}_{ti^*}^m = a_{ti}^m$. If $\varsigma_i^m > 0$ and $\varsigma_i^s > 0$, then (100) and (101) imply $\underline{a}_{ti^*}^m = \underline{a}_{ti^*}^s = 0$, and according to (104), (105) and (106), this is only possible if $a_{ti}^m = a_{ti}^s = 0$. Conditions (104) and (105) in turn imply $(\underline{a}_{tj}^m, \underline{a}_{tj}^s) = (a_{tj}^m, a_{tj}^s)$. Similarly, if $\varsigma_j^m > 0$ and $\varsigma_j^s > 0$, then (102) and (103) imply $\underline{a}_{tj}^m = \underline{a}_{tj}^s = 0$, and according to (106) this is only possible if $a_{tj}^m = a_{tj}^s = 0$. Conditions (104) and (105) in turn imply $(\underline{a}_{ti^*}^m, \underline{a}_{ti^*}^s) = (a_{ti}^m, a_{ti}^s)$. So far we have simply verified that there is no trade between investors i and j, i.e., $(\underline{a}_{ti^*}^m, \underline{a}_{ti^*}^s) = (a_{ti}^m, a_{ti}^s)$ and $(\underline{a}_{tj}^m, \underline{a}_{tj}^s) = (a_{tj}^m, a_{tj}^s)$, if $a_{ti}^m = a_{tj}^m = 0$, or $a_{ti}^s = a_{tj}^s = 0$, or

$a^m_{ti} = a^s_{ti} = 0$, or $a^m_{tj} = a^s_{tj} = 0$. Thus there are seven binding patterns for $(\varsigma^m_i, \varsigma^s_i, \varsigma^m_j, \varsigma^s_j)$ that remain to be considered.

(i) $\varsigma^m_i = \varsigma^s_i = \varsigma^m_j = \varsigma^s_j = 0$. Conditions (93)-(96) imply that this case is only possible if $\varepsilon_i = \varepsilon_j$, and conditions (104), (105) and (106), imply that the solution consists of any pair of post trade portfolios $(\underline{a}^m_{ti^*}, \underline{a}^s_{ti^*})$ and $(\underline{a}^m_{tj}, \underline{a}^s_{tj})$ that satisfy

$$\begin{aligned}
\underline{a}^m_{tj} &= a^m_{tj} - \frac{\varepsilon_j y_t + \phi^s_t}{\phi^m_t} \left(a^s_{ti} - \underline{a}^s_{ti^*}\right) \\
\underline{a}^m_{ti^*} &= a^m_{ti} + \frac{\varepsilon_j y_t + \phi^s_t}{\phi^m_t} \left(a^s_{ti} - \underline{a}^s_{ti^*}\right) \\
\underline{a}^s_{tj} &= a^s_{ti} + a^s_{tj} - \underline{a}^s_{ti^*} \\
\underline{a}^s_{ti^*} &\in \left[a^s_{ti} - \min\left(\frac{\phi^m_t}{\varepsilon_j y_t + \phi^s_t} a^m_{tj}, a^s_{ti}\right), a^s_{ti} + \min\left(\frac{\phi^m_t}{\varepsilon_j y_t + \phi^s_t} a^m_{ti}, a^s_{tj}\right)\right].
\end{aligned}$$

(ii) $\varsigma^s_i = \varsigma^m_j = \varsigma^s_j = 0 < \varsigma^m_i$. Condition (100) implies $\underline{a}^m_{ti^*} = 0$, and from (104) we obtain $\underline{a}^m_{tj} = a^m_{ti} + a^m_{tj}$. Then condition (106) yields

$$\underline{a}^s_{tj} = a^s_{tj} - \frac{\phi^m_t}{\varepsilon_j y_t + \phi^s_t} a^m_{ti}$$

and condition (105) implies

$$\underline{a}^s_{ti^*} = a^s_{ti} + \frac{\phi^m_t}{\varepsilon_j y_t + \phi^s_t} a^m_{ti}.$$

Notice that $\varsigma^s_j = 0$ requires $\underline{a}^s_{tj} \geq 0$ which is equivalent to

$$\phi^m_t a^m_{ti} \leq \left(\varepsilon_j y_t + \phi^s_t\right) a^s_{tj}.$$

Conditions (93)-(96) imply $\varsigma^m_i = (\varepsilon_i - \varepsilon_j)\, y_t \frac{\phi^m_t}{\varepsilon_j y_t + \phi^s_t}$, so $\varsigma^m_i > 0$ requires $\varepsilon_j < \varepsilon_i$.

(iii) $\varsigma^m_i = \varsigma^m_j = \varsigma^s_j = 0 < \varsigma^s_i$. Condition (101) implies $\underline{a}^s_{ti^*} = 0$, and from (105) we obtain $\underline{a}^s_{tj} = a^s_{ti} + a^s_{tj}$. Then condition (106) yields

$$\underline{a}^m_{tj} = a^m_{tj} - \frac{\varepsilon_j y_t + \phi^s_t}{\phi^m_t} a^s_{ti}$$

and condition (104) implies

$$\underline{a}^m_{ti^*} = a^m_{ti} + \frac{\varepsilon_j y_t + \phi^s_t}{\phi^m_t} a^s_{ti}.$$

Notice that $\varsigma^m_j = 0$ requires $\underline{a}^m_{tj} \geq 0$ which is equivalent to

$$\left(\varepsilon_j y_t + \phi^s_t\right) a^s_{ti} \leq \phi^m_t a^m_{tj}.$$

Conditions (93)-(96) imply $\varsigma_i^s = (\varepsilon_j - \varepsilon_i)\, y_t$, so $\varsigma_i^s > 0$ requires $\varepsilon_i < \varepsilon_j$.

(iv) $\varsigma_i^m = \varsigma_i^s = \varsigma_j^s = 0 < \varsigma_j^m$. Condition (102) implies $\underline{a}_{tj}^m = 0$, and from (104) we obtain $\underline{a}_{ti*}^m = a_{ti}^m + a_{tj}^m$. Then (105) and (106) imply

$$\underline{a}_{tj}^s = a_{tj}^s + \frac{\phi_t^m}{\varepsilon_j y_t + \phi_t^s} a_{tj}^m$$
$$\underline{a}_{ti*}^s = a_{ti}^s - \frac{\phi_t^m}{\varepsilon_j y_t + \phi_t^s} a_{tj}^m.$$

Notice that $\varsigma_i^s = 0$ requires $\underline{a}_{ti*}^s \geq 0$ which is equivalent to

$$\phi_t^m a_{tj}^m \leq (\varepsilon_j y_t + \phi_t^s)\, a_{ti}^s.$$

Conditions (93)-(96) imply $\varsigma_j^m = (\varepsilon_j - \varepsilon_i)\, y_t \frac{\phi_t^m}{\varepsilon_j y_t + \phi_t^s}$, so $\varsigma_j^m > 0$ requires $\varepsilon_i < \varepsilon_j$.

(v) $\varsigma_i^m = \varsigma_i^s = \varsigma_j^m = 0 < \varsigma_j^s$. Condition (103) implies $\underline{a}_{tj}^s = 0$, and from (105) we obtain $\underline{a}_{ti*}^s = a_{ti}^s + a_{tj}^s$. Then (104) and (106) imply

$$\underline{a}_{tj}^m = a_{tj}^m + \frac{\varepsilon_j y_t + \phi_t^s}{\phi_t^m} a_{tj}^s$$
$$\underline{a}_{ti*}^m = a_{ti}^m - \frac{\varepsilon_j y_t + \phi_t^s}{\phi_t^m} a_{tj}^s.$$

Notice that $\varsigma_i^m = 0$ requires $\underline{a}_{ti*}^m \geq 0$ which is equivalent to

$$(\varepsilon_j y_t + \phi_t^s)\, a_{tj}^s \leq \phi_t^m a_{ti}^m.$$

Conditions (93)-(96) imply $\varsigma_j^s = (\varepsilon_i - \varepsilon_j)\, y_t$, so $\varsigma_j^s > 0$ requires $\varepsilon_j < \varepsilon_i$.

(vi) $\varsigma_i^m, \varsigma_j^s \in \mathbb{R}_{++}$ and $\varsigma_i^s = \varsigma_j^m = 0$. In this case, conditions (100) and (103) give $\underline{a}_{ti*}^m = \underline{a}_{tj}^s = 0$, and (104) and (105) imply $\underline{a}_{tj}^m = a_{ti}^m + a_{tj}^m$ and $\underline{a}_{ti*}^s = a_{ti}^s + a_{tj}^s$. Condition (106) implies the following restriction must be satisfied

$$\phi_t^m a_{ti}^m = (\varepsilon_j y_t + \phi_t^s) a_{tj}^s.$$

Conditions (93)-(96) imply $\varsigma_i^m = (\rho - 1)\, \phi_t^m$ and $\varsigma_j^s = (\varepsilon_i - \varepsilon_j)\, y_t - (\rho - 1)\, (\varepsilon_j y_t + \phi_t^s)$, so $\varsigma_i^m > 0$ requires $\rho > 1$, and ς_j^s requires $\varepsilon_j < \varepsilon_i$.

(vii) $\varsigma_i^m = \varsigma_j^s = 0$ and $\varsigma_i^s, \varsigma_j^m \in \mathbb{R}_{++}$. In this case, conditions (101) and (102) give $\underline{a}_{ti*}^s = \underline{a}_{tj}^m = 0$, and (104) and (105) imply $\underline{a}_{ti*}^m = a_{ti}^m + a_{tj}^m$ and $\underline{a}_{tj}^s = a_{ti}^s + a_{tj}^s$. Condition (106) implies the following restriction must be satisfied

$$\phi_t^m a_{tj}^m = (\varepsilon_j y_t + \phi_t^s) a_{ti}^s.$$

Conditions (93)-(96) imply $\varsigma_j^m = (1-\rho)\,\phi_t^m$ and $\varsigma_i^s = (\varepsilon_j - \varepsilon_i)\,y_t - (1-\rho)\,(\varepsilon_j y_t + \phi_t^s)$, so $\varsigma_j^m > 0$ requires $\rho \in (0,1)$, and $\varsigma_i^s > 0$ requires $\varepsilon_i < \varepsilon_j$. ■

Proof of Lemma 4. (i) With Lemma 1, (10) becomes

$$\begin{aligned} V_t^D\left(a_{td}^m, a_{td}^s\right) = {} & \kappa\theta \int \bar{\phi}_t \left[\overline{a}_{td}^m + p_t \overline{a}_{td}^s - \left(a_{td}^m + p_t a_{td}^s\right)\right] dH_t\left(\boldsymbol{a}_{ti}, \varepsilon\right) \\ & + \kappa\left(1-\theta\right) \int \bar{\phi}_t \left[\overline{a}_{td^*}^m + p_t \overline{a}_{td^*}^s - \left(a_{td}^m + p_t a_{td}^s\right)\right] dH_t\left(\boldsymbol{a}_{ti}, \varepsilon\right) \\ & + \bar{\phi}_t\left(a_{td}^m + p_t a_{td}^s\right) + W_t^D\left(\mathbf{0}\right) \end{aligned}$$

where we have used the more compact notation introduced in Lemma 2, i.e., $\overline{a}_{ti^*}^k \equiv \overline{a}_{i^*}^k(\boldsymbol{a}_{ti}, \boldsymbol{a}_{td}, \varepsilon; \boldsymbol{\psi}_t)$, $\overline{a}_{td}^k \equiv \overline{a}_{d}^k(\boldsymbol{a}_{ti}, \boldsymbol{a}_{td}, \varepsilon; \boldsymbol{\psi}_t)$, $\overline{a}_{ti}^k \equiv \overline{a}_{i}^k(\boldsymbol{a}_{ti}, \boldsymbol{a}_{td}, \varepsilon; \boldsymbol{\psi}_t)$, and $\overline{a}_{td^*}^k \equiv \overline{a}_{d^*}^k(\boldsymbol{a}_{ti}, \boldsymbol{a}_{td}, \varepsilon; \boldsymbol{\psi}_t)$, for $k = m, s$. Use Corollary 1 to arrive at

$$\begin{aligned} V_t^D\left(a_{td}^m, a_{td}^s\right) = {} & \kappa\left(1-\theta\right) \int \bar{\phi}_t \left[\mathbb{I}_{\{\varepsilon < \varepsilon_t^*\}} \frac{\left(\varepsilon_t^* - \varepsilon\right) y_t}{\varepsilon_t^* y_t + \phi_t^s} p_t a_{ti}^s + \mathbb{I}_{\{\varepsilon_t^* \leq \varepsilon\}} \frac{\left(\varepsilon - \varepsilon_t^*\right) y_t}{\varepsilon y_t + \phi_t^s} a_{ti}^m\right] dH_t\left(\boldsymbol{a}_{ti}, \varepsilon\right) \\ & + \bar{\phi}_t\left(a_{td}^m + p_t a_{td}^s\right) + W_t^D\left(\mathbf{0}\right) \end{aligned}$$

where $\mathbb{I}_{\{\varepsilon < \varepsilon_t^*\}}$ is an indicator function that takes the value 1 if $\varepsilon < \varepsilon_t^*$, and 0 otherwise. To obtain (17), use the fact that $dH_t\left(\boldsymbol{a}_{ti}, \varepsilon\right) = dF_t^I\left(\boldsymbol{a}_{ti}\right) dG\left(\varepsilon\right)$.

(ii) With (63) and the notation introduced in Lemma 2 and Lemma 3, (12) becomes

$$\begin{aligned} V_t^I\left(a_{ti}^m, a_{ti}^s, \varepsilon_i\right) = {} & \delta\theta \int \left[\phi_t^m\left(\overline{a}_{ti^*}^m - a_{ti}^m\right) + \left(\varepsilon_i y_t + \phi_t^s\right)\left(\overline{a}_{ti^*}^s - a_{ti}^s\right)\right] dF_t^D\left(\boldsymbol{a}_{td}\right) \\ & + \delta\left(1-\theta\right) \int \left[\phi_t^m\left(\overline{a}_{ti}^m - \overline{a}_{ti}^m\right) + \left(\varepsilon_i y_t + \phi_t^s\right)\left(\overline{a}_{ti}^s - a_{ti}^s\right)\right] dF_t^D\left(\boldsymbol{a}_{td}\right) \\ & + \alpha \int \tilde{\eta}\left(\varepsilon_i, \varepsilon_j\right)\left[\phi_t^m\left(\underline{a}_{ti^*}^m - a_{ti}^m\right) + \left(\varepsilon_i y_t + \phi_t^s\right)\left(\underline{a}_{ti^*}^s - a_{ti}^s\right)\right] dH_t\left(\boldsymbol{a}_{tj}, \varepsilon_j\right) \\ & + \alpha \int \left[1 - \tilde{\eta}\left(\varepsilon_i, \varepsilon_j\right)\right]\left[\phi_t^m\left(\underline{a}_{ti}^m - a_{ti}^m\right) + \left(\varepsilon_i y_t + \phi_t^s\right)\left(\underline{a}_{ti}^s - a_{ti}^s\right)\right] dH_t\left(\boldsymbol{a}_{tj}, \varepsilon_j\right) \\ & + \phi_t^m a_{ti}^m + \left(\varepsilon_i y_t + \phi_t^s\right) a_{ti}^s + W_t^I\left(\mathbf{0}\right). \end{aligned}$$

Use $\tilde{\eta}\left(\varepsilon_i, \varepsilon_j\right) \equiv \eta \mathbb{I}_{\{\varepsilon_j < \varepsilon_i\}} + \left(1-\eta\right) \mathbb{I}_{\{\varepsilon_i < \varepsilon_j\}} + \left(1/2\right) \mathbb{I}_{\{\varepsilon_i = \varepsilon_j\}}$ and substitute the bargaining outcomes reported in Lemma 2 and Lemma 3 to obtain

$$
\begin{aligned}
V_t^I\left(a_{ti}^m, a_{ti}^s, \varepsilon_i\right) = {} & \delta\theta \mathbb{I}_{\{\varepsilon_t^* \le \varepsilon_i\}} \frac{\left(\varepsilon_i - \varepsilon_t^*\right) y_t}{\varepsilon_t^* y_t + \phi_t^s} \phi_t^m a_{ti}^m + \delta\theta \mathbb{I}_{\{\varepsilon_i < \varepsilon_t^*\}} \left(\varepsilon_t^* - \varepsilon_i\right) y_t a_{ti}^s \\
& + \alpha\eta \int\int \mathbb{I}_{\{\varepsilon_j \le \varepsilon_i\}} \Big[-\phi_t^m \min\left\{p_t^o(\varepsilon_j) a_{tj}^s, a_{ti}^m\right\} \\
& + \left(\varepsilon_i y_t + \phi_t^s\right) \min\left\{\tfrac{a_{ti}^m}{p_t^o(\varepsilon_j)}, a_{tj}^s\right\} \Big] dF_t^I\left(\boldsymbol{a}_{tj}\right) dG\left(\varepsilon_j\right) \\
& + \alpha\left(1-\eta\right) \int\int \mathbb{I}_{\{\varepsilon_i < \varepsilon_j\}} \Big[\phi_t^m \min\left\{p_t^o(\varepsilon_j) a_{ti}^s, a_{tj}^m\right\} \\
& - \left(\varepsilon_i y_t + \phi_t^s\right) \min\left\{\tfrac{a_{tj}^m}{p_t^o(\varepsilon_j)}, a_{ti}^s\right\} \Big] dF_t^I\left(\boldsymbol{a}_{tj}\right) dG\left(\varepsilon_j\right) \\
& + \phi_t^m a_{ti}^m + \left(\varepsilon_i y_t + \phi_t^s\right) a_{ti}^s + W_t^I\left(\mathbf{0}\right). \qquad (107)
\end{aligned}
$$

From (11), we anticipate that as in Lagos and Wright (2005), the beginning-of-period distribution of assets across investors will be degenerate, i.e., $(a_{t+1j}^m, a_{t+1j}^s) = (A_{It+1}^m, A_{It+1}^s)$ for all $j \in \mathcal{I}$, so (107) can be written as (18). ■

Proof of Lemma 5. With Lemma 4, the dealer's problem in the second subperiod of period t, (14), becomes

$$
W_t^D\left(\mathbf{0}\right) = \max_{\tilde{\boldsymbol{a}}_{t+1} \in \mathbb{R}_+^2} \left[(-\phi_t^m + \beta \mathbb{E}_t \bar{\phi}_{t+1}) \tilde{a}_{t+1}^m + (-\phi_t^s + \beta\pi \mathbb{E}_t \bar{\phi}_{t+1} p_{t+1}) \tilde{a}_{t+1}^s \right] + \beta \mathbb{E}_t V_{t+1}^D\left(\mathbf{0}\right). \quad (108)
$$

From (18),

$$
\begin{aligned}
\int V_{t+1}^I\left(a_{t+1}^m, a_{t+1}^s, \varepsilon_i\right) dG\left(\varepsilon_i\right) = {} & \phi_{t+1}^m a_{t+1}^m + \int \left(\varepsilon_i y_{t+1} + \phi_{t+1}^s\right) a_{t+1}^s dG\left(\varepsilon_i\right) + W_{t+1}^I\left(\mathbf{0}\right) \\
& + \delta\theta \int \mathbb{I}_{\{\varepsilon_{t+1}^* \le \varepsilon_i\}} \frac{\left(\varepsilon_i - \varepsilon_{t+1}^*\right) y_{t+1}}{\varepsilon_{t+1}^* y_{t+1} + \phi_{t+1}^s} \phi_{t+1}^m a_{t+1}^m dG\left(\varepsilon_i\right) \\
& + \delta\theta \int \mathbb{I}_{\{\varepsilon_i < \varepsilon_{t+1}^*\}} \left(\varepsilon_{t+1}^* - \varepsilon_i\right) y_{t+1} a_{t+1}^s dG\left(\varepsilon_i\right) \\
& + \alpha\eta \int_{\left[\frac{\phi_{t+1}^m a_{t+1}^m}{A_{It+1}^s} - \phi_{t+1}^s\right] \frac{1}{y_{t+1}}} \int_{\varepsilon_j} \frac{\left(\varepsilon_i - \varepsilon_j\right) y_{t+1}}{\varepsilon_j y_{t+1} + \phi_{t+1}^s} \phi_{t+1}^m a_{t+1}^m dG\left(\varepsilon_i\right) dG\left(\varepsilon_j\right) \\
& + \alpha\eta \int^{\left[\frac{\phi_{t+1}^m a_{t+1}^m}{A_{It+1}^s} - \phi_{t+1}^s\right] \frac{1}{y_{t+1}}} \int_{\varepsilon_j} \left(\varepsilon_i - \varepsilon_j\right) y_{t+1} A_{It+1}^s dG\left(\varepsilon_i\right) dG\left(\varepsilon_j\right) \\
& + \alpha\left(1-\eta\right) \int_{\left[\frac{\phi_{t+1}^m A_{It+1}^m}{a_{t+1}^s} - \phi_{t+1}^s\right] \frac{1}{y_{t+1}}} \int^{\varepsilon_j} \frac{\left(\varepsilon_j - \varepsilon_i\right) y_{t+1}}{\varepsilon_j y_{t+1} + \phi_{t+1}^s} \phi_{t+1}^m A_{It+1}^m dG\left(\varepsilon_i\right) dG\left(\varepsilon_j\right) \\
& + \alpha\left(1-\eta\right) \int^{\left[\frac{\phi_{t+1}^m A_{It+1}^m}{a_{t+1}^s} - \phi_{t+1}^s\right] \frac{1}{y_{t+1}}} \int^{\varepsilon_j} \left(\varepsilon_j - \varepsilon_i\right) y_{t+1} a_{t+1}^s dG\left(\varepsilon_i\right) dG\left(\varepsilon_j\right)
\end{aligned}
$$

so the investor's problem (11) can be written as in (63), with

$$W_t^I(\mathbf{0}) = \max_{\tilde{a}_{t+1}^m \in \mathbb{R}_+} \Bigg\{ -\phi_t^m \tilde{a}_{t+1}^m + \beta \mathbb{E}_t \Bigg[\Bigg(1 + \delta\theta \int \mathbb{I}_{\{\varepsilon_{t+1}^* \leq \varepsilon_i\}} \frac{(\varepsilon_i - \varepsilon_{t+1}^*) y_{t+1}}{\varepsilon_{t+1}^* y_{t+1} + \phi_{t+1}^s} dG(\varepsilon_i)$$

$$+ \alpha\eta \int_{\left[\frac{\phi_{t+1}^m a_{t+1}^m}{A_{It+1}^s} - \phi_{t+1}^s\right]\frac{1}{y_{t+1}}} \int_{\varepsilon_j} \frac{(\varepsilon_i - \varepsilon_j) y_{t+1}}{\varepsilon_j y_{t+1} + \phi_{t+1}^s} dG(\varepsilon_i)\, dG(\varepsilon_j) \Bigg) \phi_{t+1}^m \tilde{a}_{t+1}^m$$

$$+ \alpha\eta \int^{\left[\frac{\phi_{t+1}^m \tilde{a}_{t+1}^m}{A_{It+1}^s} - \phi_{t+1}^s\right]\frac{1}{y_{t+1}}} \int_{\varepsilon_j} (\varepsilon_i - \varepsilon_j) y_{t+1} dG(\varepsilon_i)\, dG(\varepsilon_j)\, A_{It+1}^s \Bigg] \Bigg\}$$

$$+ \max_{\tilde{a}_{t+1}^s \in \mathbb{R}_+} \Bigg\{ -\phi_t^s \tilde{a}_{t+1}^s + \beta \mathbb{E}_t \Bigg[\Bigg(\int \left(\varepsilon_i y_{t+1} + \phi_{t+1}^s\right) dG(\varepsilon_i)$$

$$+ \delta\theta \int \mathbb{I}_{\{\varepsilon_i < \varepsilon_{t+1}^*\}} \left(\varepsilon_{t+1}^* - \varepsilon_i\right) y_{t+1} dG(\varepsilon_i)$$

$$+ \alpha(1-\eta) \int^{\left[\frac{\phi_{t+1}^m A_{It+1}^m}{a_{t+1}^s} - \phi_{t+1}^s\right]\frac{1}{y_{t+1}}} \int^{\varepsilon_j} (\varepsilon_j - \varepsilon_i) y_{t+1} dG(\varepsilon_i)\, dG(\varepsilon_j) \Bigg) a_{t+1}^s$$

$$+ \alpha(1-\eta) \int_{\left[\frac{\phi_{t+1}^m A_{It+1}^m}{a_{t+1}^s} - \phi_{t+1}^s\right]\frac{1}{y_{t+1}}} \int^{\varepsilon_j} \frac{(\varepsilon_j - \varepsilon_i) y_{t+1}}{\varepsilon_j y_{t+1} + \phi_{t+1}^s} dG(\varepsilon_i)\, dG(\varepsilon_j)\, \phi_{t+1}^m A_{It+1}^m \Bigg] \Bigg\}$$

$$+ T_t + \beta \mathbb{E}_t W_{t+1}^I(\mathbf{0}), \tag{109}$$

where $a_{t+1}^s = \pi \tilde{a}_{t+1}^s + (1-\pi) A^s$. The first-order necessary and sufficient conditions for optimization of (108) are (19) and (20). The first-order necessary and sufficient conditions for optimization of (109) are (21) and (22). ■

Proof of Proposition 2. In a stationary equilibrium, the dealer's Euler equations in Lemma 5 become

$$\mu \geq \bar{\beta}, \text{ “ = ” if } \tilde{a}_{t+1d}^m > 0$$

$$\phi^s \geq \frac{\bar{\beta}\pi}{1 - \bar{\beta}\pi} \varepsilon^*, \text{ “ = ” if } \tilde{a}_{t+1d}^s > 0.$$

The maintained assumption $\mu > \bar{\beta}$ implies $\tilde{a}_{t+1d}^m = 0$. Similarly, in a stationary monetary equilibrium the investor's Euler equations in Lemma 5 become

$$\mu = \bar{\beta} \Bigg[1 + \delta\theta \int_{\varepsilon^*}^{\varepsilon_H} \frac{\varepsilon_i - \varepsilon^*}{\varepsilon^* + \phi^s} dG(\varepsilon_i)$$

$$+ \alpha\eta \int_{\varepsilon^c}^{\varepsilon_H} \int_{\varepsilon_j}^{\varepsilon_H} \frac{\varepsilon_i - \varepsilon_j}{\varepsilon_j + \phi^s} dG(\varepsilon_i)\, dG(\varepsilon_j) \Bigg] \tag{110}$$

$$\phi^s \geq \frac{\bar{\beta}\pi}{1-\bar{\beta}\pi}\left[\bar{\varepsilon}+\delta\theta\int_{\varepsilon_L}^{\varepsilon^*}(\varepsilon^*-\varepsilon_i)dG(\varepsilon_i)+\alpha(1-\eta)\varphi(\varepsilon^c)\right]$$

where $\varepsilon^c \equiv Z/A_I^s - \phi^s$, $\varphi(\varepsilon) \equiv \int_{\varepsilon_L}^{\varepsilon}\int_{\varepsilon_L}^{\varepsilon_j}(\varepsilon_j-\varepsilon_i)\,dG(\varepsilon_i)\,dG(\varepsilon_j)$, and the second condition holds with "=" if $\tilde{a}_{t+1i}^s > 0$. Together, the dealer's and the investor's Euler equations for equity imply

$$\phi^s = \frac{\bar{\beta}\pi}{1-\bar{\beta}\pi}\max\left\{\varepsilon^*, \bar{\varepsilon}+\delta\theta\int_{\varepsilon_L}^{\varepsilon^*}(\varepsilon^*-\varepsilon_i)dG(\varepsilon_i)+\alpha(1-\eta)\varphi(\varepsilon^c)\right\}. \tag{111}$$

As $\mu \to \bar{\beta}$, (110) implies

$$\delta\theta\int_{\varepsilon^*}^{\varepsilon_H}\frac{\varepsilon_i-\varepsilon^*}{\varepsilon^*+\phi^s}dG(\varepsilon_i)+\alpha\eta\int_{\varepsilon^c}^{\varepsilon_H}\int_{\varepsilon_j}^{\varepsilon_H}\frac{\varepsilon_i-\varepsilon_j}{\varepsilon_j+\phi^s}dG(\varepsilon_i)\,dG(\varepsilon_j)\to 0,$$

a condition that can only hold if $\varepsilon^* \to \varepsilon_H$ and $\varepsilon^c \to \varepsilon_H$. The fact that $\varepsilon^* \to \varepsilon_H$ means that among investors who contact dealers, only those with preference type ε_H purchase equity. The fact that $\varepsilon^c \to \varepsilon_H$ implies that in bilateral trades between investors, the investor with the higher valuation purchases all his counterparty's equity holdings (the investor who wishes to buy is never constrained by his real money balances as $\mu \to \bar{\beta}$). Finally, as $\mu \to \bar{\beta}$,

$$\phi^s \to \frac{\bar{\beta}\pi}{1-\bar{\beta}\pi}\max\{\varepsilon_H, \bar{\varepsilon}+\delta\theta(\varepsilon_H-\bar{\varepsilon})+\alpha(1-\eta)\varphi(\varepsilon_H)\} = \frac{\bar{\beta}\pi}{1-\bar{\beta}\pi}\varepsilon_H,$$

so $\tilde{a}_{t+1d}^s = A^s$ for all t, i.e., only dealers hold equity overnight. ■

Lemma 6 *Consider $\hat{\mu}$ and $\bar{\mu}$ as defined in (24). Then $\hat{\mu} < \bar{\mu}$.*

Proof of Lemma 6. Define $\Upsilon(\zeta): \mathbb{R} \to \mathbb{R}$ by $\Upsilon(\zeta) \equiv \bar{\beta}\left[1+\delta\theta(1-\bar{\beta}\pi)\zeta\right]$. Let $\hat{\zeta} \equiv \frac{(1-\delta\theta)(\hat{\varepsilon}-\bar{\varepsilon})}{\delta\theta\hat{\varepsilon}}$ and $\bar{\zeta} \equiv \frac{\bar{\varepsilon}-\varepsilon_L}{\beta\pi\bar{\varepsilon}+(1-\beta\pi)\varepsilon_L}$, so that $\hat{\mu} = \Upsilon(\hat{\zeta})$ and $\bar{\mu} = \Upsilon(\bar{\zeta})$. Since Υ is strictly increasing, $\hat{\mu} < \bar{\mu}$ if and only if $\hat{\zeta} < \bar{\zeta}$. With (25) and the fact that $\bar{\varepsilon} \equiv \int_{\varepsilon_L}^{\varepsilon_H}\varepsilon dG(\varepsilon) = \varepsilon_H - \int_{\varepsilon_L}^{\varepsilon_H}G(\varepsilon)\,d\varepsilon$,

$$\hat{\zeta} = \frac{\int_{\hat{\varepsilon}}^{\varepsilon_H}[1-G(\varepsilon)]\,d\varepsilon}{\bar{\varepsilon}+\delta\theta\int_{\varepsilon_L}^{\hat{\varepsilon}}G(\varepsilon)\,d\varepsilon},$$

so clearly,

$$\hat{\zeta} < \frac{\int_{\varepsilon_L}^{\varepsilon_H}[1-G(\varepsilon)]\,d\varepsilon}{\bar{\varepsilon}} = \frac{\bar{\varepsilon}-\varepsilon_L}{\bar{\varepsilon}} < \bar{\zeta}.$$

Hence $\hat{\mu} < \bar{\mu}$. ■

Lemma 7 *In a stationary equilibrium, the interdealer market clearing condition* $\bar{A}^s_{Dt} + \bar{A}^s_{It} = A^s_{Dt} + \delta A^s_{It}$ *is equivalent to*

$$\delta\theta\left[1-G(\varepsilon^*)\right]\left(A^s_I + \frac{Z}{\varepsilon^*+\phi^s}\right) + \delta\left(1-\theta\right)\int_{\varepsilon^*}^{\varepsilon_H}\left[A^s_I + \frac{Z}{\varepsilon+\phi^s}\right]dG\left(\varepsilon\right) = A^s_D + \delta A^s_I. \quad (112)$$

Proof of Lemma 7. Use $\delta = \kappa v$ in $\bar{A}^s_{Dt} + \bar{A}^s_{It} = A^s_{Dt} + \delta A^s_{It}$ to obtain

$$\begin{aligned}&\theta\int\left\{\hat{a}^s_d\left[\bar{\boldsymbol{a}}_d\left(\boldsymbol{a}_{ti},\boldsymbol{a}_{td},\varepsilon;\boldsymbol{\psi}_t\right);\boldsymbol{\psi}_t\right] + \overline{a}^s_{i^*}(\boldsymbol{a}_{ti},\boldsymbol{a}_{td},\varepsilon;\boldsymbol{\psi}_t)\right\}dF^D_t\left(\boldsymbol{a}_{td}\right)dF^I_t\left(\boldsymbol{a}_{ti}\right)dG\left(\varepsilon\right)\\&+\left(1-\theta\right)\int\left\{\hat{a}^s_d\left[\bar{\boldsymbol{a}}_{d^*}\left(\boldsymbol{a}_{ti},\boldsymbol{a}_{td},\varepsilon;\boldsymbol{\psi}_t\right);\boldsymbol{\psi}_t\right] + \overline{a}^s_{i}(\boldsymbol{a}_{ti},\boldsymbol{a}_{td},\varepsilon;\boldsymbol{\psi}_t)\right\}dF^D_t\left(\boldsymbol{a}_{td}\right)dF^I_t\left(\boldsymbol{a}_{ti}\right)dG\left(\varepsilon\right)\\&=\int a^s_{td}dF^D_t\left(\boldsymbol{a}_{td}\right) + \int a^s_{ti}dF^I_t\left(\boldsymbol{a}_{ti}\right) + \frac{\left(1-\kappa\right)v}{\delta}\int\left[a^s_{td} - \hat{a}^s_d(\boldsymbol{a}_{td};\boldsymbol{\psi}_t)\right]dF^D_t\left(\boldsymbol{a}_{td}\right). \quad (113)\end{aligned}$$

Since $\phi^s_t = \phi^s y_t < \varepsilon^* y_t + \phi^s y_t = \bar{\phi}^s y_t \equiv p_t\phi^m_t$ in a stationary equilibrium, Lemma 1 implies

$$\hat{a}^s_d\left[\bar{\boldsymbol{a}}_d\left(\boldsymbol{a}_{ti},\boldsymbol{a}_{td},\varepsilon;\boldsymbol{\psi}_t\right);\boldsymbol{\psi}_t\right] = \hat{a}^s_d\left[\bar{\boldsymbol{a}}_{d^*}\left(\boldsymbol{a}_{ti},\boldsymbol{a}_{td},\varepsilon;\boldsymbol{\psi}_t\right);\boldsymbol{\psi}_t\right] = \hat{a}^s_d(\boldsymbol{a}_{td};\boldsymbol{\psi}_t) = 0. \quad (114)$$

With (114) and the fact that $\int a^s_{ti}dF^I_t\left(\boldsymbol{a}_{ti}\right) = A^s_I$ and $v\int a^s_{td}dF^D_t\left(\boldsymbol{a}_{td}\right) = A^s_D$, (113) becomes

$$\begin{aligned}A^s_D + \delta A^s_I = \delta\theta&\int\overline{a}^s_{i^*}(\boldsymbol{a}_{ti},\boldsymbol{a}_{td},\varepsilon;\boldsymbol{\psi}_t)dF^D_t\left(\boldsymbol{a}_{td}\right)dF^I_t\left(\boldsymbol{a}_{ti}\right)dG\left(\varepsilon\right)\\&+\delta\left(1-\theta\right)\int\overline{a}^s_{i}(\boldsymbol{a}_{ti},\boldsymbol{a}_{td},\varepsilon;\boldsymbol{\psi}_t)dF^D_t\left(\boldsymbol{a}_{td}\right)dF^I_t\left(\boldsymbol{a}_{ti}\right)dG\left(\varepsilon\right). \quad (115)\end{aligned}$$

From Lemma 2,

$$\begin{aligned}\overline{a}^s_{i^*}(\boldsymbol{a}_{ti},\boldsymbol{a}_{td},\varepsilon;\boldsymbol{\psi}_t) &= \mathbb{I}_{\{\varepsilon^*<\varepsilon\}}\left(a^s_{ti} + \frac{1}{p_t}a^m_{ti}\right) + \mathbb{I}_{\{\varepsilon=\varepsilon^*\}}\overline{a}^s_{i^*}\\ \overline{a}^s_{i}(\boldsymbol{a}_{ti},\boldsymbol{a}_{td},\varepsilon;\boldsymbol{\psi}_t) &= \mathbb{I}_{\{\varepsilon^*<\varepsilon\}}\left[a^s_{ti} + \left(\frac{\varepsilon^*+\phi^s}{\varepsilon+\phi^s}\right)\frac{1}{p_t}a^m_{ti}\right] + \mathbb{I}_{\{\varepsilon=\varepsilon^*\}}\overline{a}^s_{i}\end{aligned}$$

where $\overline{a}^s_{i^*}, \overline{a}^s_i \in [0, a^s_{ti} + a^m_{ti}/p_t]$, so (115) becomes

$$\delta\theta\left[1-G(\varepsilon^*)\right]\left(A^s_I + \frac{1}{p_t}A^m_{It}\right) + \delta\left(1-\theta\right)\int_{\varepsilon^*}^{\varepsilon_H}\left[A^s_I + \left(\frac{\varepsilon^*+\phi^s}{\varepsilon+\phi^s}\right)\frac{1}{p_t}A^m_{It}\right]dG\left(\varepsilon\right) = A^s_D + \delta A^s_I.$$

Finally, use the fact that in a stationary equilibrium, $\phi^m_t A^m_{It} = Zy_t$ and $p_t\phi^m_t = \bar{\phi}^s y_t = \left(\varepsilon^*+\phi^s\right)y_t$, to arrive at the expression in the statement of the lemma. ■

Proof of Proposition 3. In an equilibrium with no money (or no valued money), there is no trade in the OTC market. The first-order conditions for a dealer d and an investor i in the time-t Walrasian market are

$$\phi_t^s \geq \beta\pi\mathbb{E}_t\phi_{t+1}^s, \text{ “ = ” if } \tilde{a}_{t+1d}^s > 0$$
$$\phi_t^s \geq \beta\pi\mathbb{E}_t\left(\bar{\varepsilon}y_{t+1} + \phi_{t+1}^s\right), \text{ “ = ” if } \tilde{a}_{t+1i}^s > 0.$$

In a stationary equilibrium, $\mathbb{E}_t(\phi_{t+1}^s/\phi_t^s) = \bar{\gamma}$, and $\beta\bar{\gamma}\pi < 1$ is a maintained assumption, so no dealer holds equity. The Walrasian market for equity can only clear if $\phi^s = \frac{\bar{\beta}\pi}{1-\bar{\beta}\pi}\bar{\varepsilon}$. This establishes parts (i) and (iii) in the statement of the proposition.

Next, we turn to monetary equilibria. With $\alpha = 0$, in a stationary equilibrium (19)-(22) become

$$\mu \geq \bar{\beta}, \text{ “ = ” if } \tilde{a}_{t+1d}^m > 0 \tag{116}$$
$$\phi^s \geq \bar{\beta}\pi\bar{\phi}^s, \text{ “ = ” if } \tilde{a}_{t+1d}^s > 0 \tag{117}$$
$$1 \geq \frac{\bar{\beta}}{\mu}\left[1 + \delta\theta\frac{\int_{\varepsilon^*}^{\varepsilon_H}\left[1 - G(\varepsilon)\right]d\varepsilon}{\varepsilon^* + \phi^s}\right], \text{ “ = ” if } \tilde{a}_{t+1i}^m > 0 \tag{118}$$
$$\phi^s \geq \frac{\bar{\beta}\pi}{1-\bar{\beta}\pi}\left[\bar{\varepsilon} + \delta\theta\int_{\varepsilon_L}^{\varepsilon^*}G(\varepsilon)\,d\varepsilon\right], \text{ “ = ” if } \tilde{a}_{t+1i}^s > 0. \tag{119}$$

(In (116) we have used the fact that $\bar{\phi}^s = \varepsilon^* + \phi^s > \phi^s$.) Under our maintained assumption $\bar{\beta} < \mu$, (116) implies $\tilde{a}_{t+1d}^m = Z_D = 0$, so (118) must hold with equality for some investor in a monetary equilibrium. Thus, in order to find a monetary equilibrium there are three possible equilibrium configurations to consider depending on the binding patterns of the complementary slackness conditions (117) and (119). The market-clearing condition, $\bar{A}_{Dt}^s + \bar{A}_{It}^s = A_{Dt}^s + \delta A_{It}^s$ must hold for all three configurations. Lemma 7 shows that this condition can be written as (112) and this condition can be rearranged to deliver (31). The rest of the proof proceeds in three steps.

Step 1: Try to construct a stationary monetary equilibrium with $\tilde{a}_{t+1d}^s = 0$ for all $d \in \mathcal{D}$, and $\tilde{a}_{t+1i}^s > 0$ for some $i \in \mathcal{I}$. The equilibrium conditions for this case are (112) together with

$$\phi^s > \bar{\beta}\pi\bar{\phi}^s \tag{120}$$
$$1 = \frac{\bar{\beta}}{\mu}\left[1 + \delta\theta\frac{\int_{\varepsilon^*}^{\varepsilon_H}\left[1 - G(\varepsilon)\right]d\varepsilon}{\varepsilon^* + \phi^s}\right] \tag{121}$$
$$\phi^s = \frac{\bar{\beta}\pi}{1-\bar{\beta}\pi}\left[\bar{\varepsilon} + \delta\theta\int_{\varepsilon_L}^{\varepsilon^*}G(\varepsilon)\,d\varepsilon\right] \tag{122}$$

and

$$\tilde{a}^m_{t+1d} = 0 \text{ for all } d \in \mathcal{D} \tag{123}$$

$$\tilde{a}^m_{t+1i} \geq 0, \text{ with " > " for some } i \in \mathcal{I} \tag{124}$$

$$\tilde{a}^s_{t+1d} = 0 \text{ for all } d \in \mathcal{D} \tag{125}$$

$$\tilde{a}^s_{t+1i} \geq 0, \text{ with " > " for some } i \in \mathcal{I}. \tag{126}$$

Conditions (121) and (122) are to be solved for the two unknowns ε^* and ϕ^s. Substitute (122) into (121) to obtain

$$1 = \frac{\bar{\beta}}{\mu}\left[1 + \delta\theta \frac{\int_{\varepsilon^*}^{\varepsilon_H}\left[1 - G(\varepsilon)\right]d\varepsilon}{\varepsilon^* + \frac{\bar{\beta}\pi}{1-\bar{\beta}\pi}\left[\bar{\varepsilon} + \delta\theta\int_{\varepsilon_L}^{\varepsilon^*} G(\varepsilon)\,d\varepsilon\right]}\right] \tag{127}$$

which is a single equation in ε^*. Define

$$T(x) \equiv \frac{\int_{x}^{\varepsilon_H}\left[1 - G(\varepsilon)\right]d\varepsilon}{\frac{1}{1-\bar{\beta}\pi}x + \frac{\bar{\beta}\pi}{1-\bar{\beta}\pi}\hat{T}(x)} - \frac{\mu - \bar{\beta}}{\bar{\beta}\delta\theta} \tag{128}$$

with

$$\hat{T}(x) \equiv \bar{\varepsilon} - x + \delta\theta\int_{\varepsilon_L}^{x} G(\varepsilon)\,d\varepsilon, \tag{129}$$

and notice that ε^* solves (127) if and only if it satisfies $T(\varepsilon^*) = 0$. T is a continuous real-valued function on $[\varepsilon_L, \varepsilon_H]$, with

$$T(\varepsilon_L) = \frac{\bar{\varepsilon} - \varepsilon_L}{\varepsilon_L + \frac{\bar{\beta}\pi}{1-\bar{\beta}\pi}\bar{\varepsilon}} - \frac{\mu - \bar{\beta}}{\bar{\beta}\delta\theta},$$

$$T(\varepsilon_H) = -\frac{\mu - \bar{\beta}}{\bar{\beta}\delta\theta} < 0,$$

and

$$T'(x) = -\frac{\left[1-G(x)\right]\left\{x + \frac{\bar{\beta}\pi}{1-\bar{\beta}\pi}\left[\bar{\varepsilon} + \delta\theta\int_{\varepsilon_L}^{x} G(\varepsilon)d\varepsilon\right]\right\} + \left[\int_{x}^{\varepsilon_H}\left[1-G(\varepsilon)\right]d\varepsilon\right]\left\{1 + \frac{\bar{\beta}\pi}{1-\bar{\beta}\pi}\delta\theta G(x)\right\}}{\left\{x + \frac{\bar{\beta}\pi}{1-\bar{\beta}\pi}\left[\bar{\varepsilon} + \delta\theta\int_{\varepsilon_L}^{x} G(\varepsilon)d\varepsilon\right]\right\}^2} < 0.$$

Hence if $T(\varepsilon_L) > 0$, or equivalently, if $\mu < \bar{\mu}$ (with $\bar{\mu}$ is as defined in (24)) then there exists a unique $\varepsilon^* \in (\varepsilon_L, \varepsilon_H)$ that satisfies $T(\varepsilon^*) = 0$ (and $\varepsilon^* \downarrow \varepsilon_L$ as $\mu \uparrow \bar{\mu}$). Once we know ε^*, ϕ^s is given by (122). Given ε^* and ϕ^s, the values of Z, $\bar{\phi}^s$, ϕ^m_t and p_t are obtained using (31) (with $A^s_I = A^s$ and $A^s_D = 0$), (28), (29) and (30). To conclude this step, notice that for this case to be an equilibrium (120) must hold, or equivalently, using $\bar{\phi}^s = \varepsilon^* + \phi^s$ and (122), it must be that $\hat{T}(\varepsilon^*) > 0$, where $\hat{T}$ is the continuous function on $[\varepsilon_L, \varepsilon_H]$ defined in (129). Notice that $\hat{T}'(x) = -[1-\delta\theta G(x)] < 0$, and $\hat{T}(\varepsilon_H) = -(1-\delta\theta)(\varepsilon_H - \bar{\varepsilon}) < 0 < \bar{\varepsilon} - \varepsilon_L = \hat{T}(\varepsilon_L)$, so there exists a unique $\hat{\varepsilon} \in (\varepsilon_L, \varepsilon_H)$ such that $\hat{T}(\hat{\varepsilon}) = 0$. (Since $\hat{T}(\bar{\varepsilon}) > 0$, and $\hat{T}' < 0$, it follows that $\bar{\varepsilon} < \hat{\varepsilon}$.) Then $\hat{T}'(x) < 0$ implies $\hat{T}(\varepsilon^*) \geq 0$ if and only if $\varepsilon^* \leq \hat{\varepsilon}$, with "=" for $\varepsilon^* = \hat{\varepsilon}$. With (128), we know that $\varepsilon^* < \hat{\varepsilon}$ if and only if $T(\hat{\varepsilon}) < 0 = T(\varepsilon^*)$, i.e., if and only if

$$\bar{\beta}\left[1 + \frac{\delta\theta\left(1-\bar{\beta}\pi\right)\int_{\hat{\varepsilon}}^{\varepsilon_H}[1-G(\varepsilon)]\,d\varepsilon}{\bar{\beta}\pi\bar{\varepsilon} + \left(1-\bar{\beta}\pi\right)\hat{\varepsilon} + \bar{\beta}\pi\delta\theta\int_{\varepsilon_L}^{\hat{\varepsilon}} G(\varepsilon)\,d\varepsilon}\right] < \mu.$$

Since $\hat{T}(\hat{\varepsilon}) = (1-\delta\theta)(\bar{\varepsilon} - \hat{\varepsilon}) + \delta\theta\int_{\hat{\varepsilon}}^{\varepsilon_H}[1-G(\varepsilon)]\,d\varepsilon = 0$, this last condition is equivalent to $\hat{\mu} < \mu$, where $\hat{\mu}$ is as defined in (24). The allocations and asset prices described in this step correspond to those in the statement of the proposition for the case with $\mu \in (\hat{\mu}, \bar{\mu})$.

Step 2: Try to construct a stationary monetary equilibrium with $a^s_{t+1d} > 0$ for some $d \in \mathcal{D}$, and $a^s_{t+1i} = 0$ for all $i \in \mathcal{I}$. The equilibrium conditions are (112), (121), (123), (124), together with

$$\phi^s = \bar{\beta}\pi\bar{\phi}^s \tag{130}$$

$$\phi^s > \frac{\bar{\beta}\pi}{1-\bar{\beta}\pi}\left[\bar{\varepsilon} + \delta\theta\int_{\varepsilon_L}^{\varepsilon^*} G(\varepsilon)\,d\varepsilon\right], \text{ “ = ” if } \tilde{a}^s_{t+1i} > 0. \tag{131}$$

$$\tilde{a}^s_{t+1d} \geq 0, \text{ with “ > ” for some } d \in \mathcal{D} \tag{132}$$

$$\tilde{a}^s_{t+1i} = 0, \text{ for all } i \in \mathcal{I}. \tag{133}$$

The conditions (121) and (130) are to be solved for ε^* and ϕ^s. First use $\bar{\phi}^s = \varepsilon^* + \phi^s$ in (130) to obtain

$$\phi^s = \frac{\bar{\beta}\pi}{1-\bar{\beta}\pi}\varepsilon^*. \tag{134}$$

Substitute (134) in (121) to obtain

$$1 = \frac{\bar{\beta}}{\mu}\left[1 + \frac{\delta\theta\left(1-\bar{\beta}\pi\right)\int_{\varepsilon^*}^{\varepsilon_H}[1-G(\varepsilon)]\,d\varepsilon}{\varepsilon^*}\right] \tag{135}$$

which is a single equation in ε^*. Define

$$R(x) \equiv \frac{\left(1-\bar{\beta}\pi\right)\int_{x}^{\varepsilon_H}[1-G(\varepsilon)]\,d\varepsilon}{x} - \frac{\mu-\bar{\beta}}{\bar{\beta}\delta\theta} \tag{136}$$

and notice that ε^* solves (135) if and only if it satisfies $R(\varepsilon^*) = 0$. R is a continuous real-valued function on $[\varepsilon_L, \varepsilon_H]$, with

$$R(\varepsilon_L) = \frac{\left(1-\bar{\beta}\pi\right)(\bar{\varepsilon}-\varepsilon_L)}{\varepsilon_L} - \frac{\mu-\bar{\beta}}{\bar{\beta}\delta\theta}$$
$$R(\varepsilon_H) = -\frac{\mu-\bar{\beta}}{\bar{\beta}\delta\theta}$$

and

$$R'(x) = -\frac{[1-G(x)]\,x + \int_x^{\varepsilon_H}[1-G(\varepsilon)]\,d\varepsilon}{\frac{1}{1-\bar{\beta}\pi}x^2} < 0.$$

Hence if $R(\varepsilon_L) > 0$, or equivalently, if

$$\mu < \bar{\beta}\left[1 + \frac{\delta\theta\left(1-\bar{\beta}\pi\right)(\bar{\varepsilon}-\varepsilon_L)}{\varepsilon_L}\right] \equiv \mu^o$$

then there exists a unique $\varepsilon^* \in (\varepsilon_L, \varepsilon_H)$ that satisfies $R(\varepsilon^*) = 0$ (and $\varepsilon^* \downarrow \varepsilon_L$ as $\mu \uparrow \mu^o$). Having solved for ε^*, ϕ^s is obtained from (134). Given ε^* and ϕ^s, the values of Z, $\bar{\phi}^s$, ϕ_t^m and p_t are obtained using (31) (with $A_D^s = A^s - A_I^s = \pi A^s$), (28), (29) and (30). Notice that for this case to be an equilibrium (131) must hold, or equivalently, using (134), it must be that $\hat{T}(\varepsilon^*) < 0$, which is in turn equivalent to $\hat{\varepsilon} < \varepsilon^*$. With (136), we know that $\hat{\varepsilon} < \varepsilon^*$ if and only if $R(\varepsilon^*) = 0 < R(\hat{\varepsilon})$, i.e., if and only if

$$\mu < \bar{\beta}\left[1 + \frac{\delta\theta\left(1-\bar{\beta}\pi\right)\int_{\hat{\varepsilon}}^{\varepsilon_H}[1-G(\varepsilon)]\,d\varepsilon}{\hat{\varepsilon}}\right],$$

which using the fact that $\hat{T}(\hat{\varepsilon}) = 0$, can be written as $\mu < \hat{\mu}$. To summarize, the prices and allocations constructed in this step constitute a stationary monetary equilibrium provided $\mu \in (\bar{\beta}, \min(\hat{\mu}, \mu^o))$. To conclude this step, we show that $\hat{\mu} < \bar{\mu} < \mu^o$, which together with the previous step will mean that there is no stationary monetary equilibrium for $\mu \geq \bar{\mu}$ (thus establishing part (*ii*) in the statement of the proposition). It is clear that $\bar{\mu} < \mu^o$, and we know that $\hat{\mu} < \bar{\mu}$ from Lemma 6. Therefore the allocations and asset prices described in this step correspond to those in the statement of the proposition for the case with $\mu \in (\bar{\beta}, \min(\hat{\mu}, \mu^o)) = (\bar{\beta}, \hat{\mu})$.

Step 3: Try to construct a stationary monetary equilibrium with $\tilde{a}_{t+1d}^s > 0$ for some $d \in \mathcal{D}$, and $\tilde{a}_{t+1i}^s > 0$ for some $i \in \mathcal{I}$. The equilibrium conditions are (112), (121), (122), (123), (124), and (130) with

$$\tilde{a}_{t+1i}^s \geq 0 \text{ and } \tilde{a}_{t+1d}^s \geq 0, \text{ with " > " for some } i \in \mathcal{I} \text{ or some } d \in \mathcal{I}.$$

Notice that ε^* and ϕ^s are obtained as in Step 2. Now, however, (122) must also hold, which together with (134) implies that

$$0 = \bar{\varepsilon} - \varepsilon^* + \delta\theta \int_{\varepsilon_L}^{\varepsilon^*} G(\varepsilon)\, d\varepsilon$$

or equivalently, (since the right side is just $\hat{T}(\varepsilon^*)$), that $\varepsilon^* = \hat{\varepsilon}$. In other words, this condition requires $R(\hat{\varepsilon}) = \hat{T}(\hat{\varepsilon})$, or equivalently, we must have $\mu = \hat{\mu}$. As before, the market-clearing condition (31) is used to obtain Z, while (28), (29), and (30) imply $\bar{\phi}^s$, ϕ_t^m, and p_t, respectively. The allocations and asset prices described in this step correspond to those in the statement of the proposition for the case with $\mu = \hat{\mu}$.

Combined, Steps 1-3 prove part (iv) in the statement of the proposition. Part $(v)(a)$ is immediate from (122) and (128), and part $(v)(b)$ from (134) and (136). ■

Corollary 2 *The marginal type, ε^*, characterized in Proposition 3 is strictly decreasing in the rate of inflation, i.e.,$\frac{\partial \varepsilon^*}{\partial \mu} < 0$ both for $\mu \in (\bar{\beta}, \hat{\mu})$, and for $\mu \in (\hat{\mu}, \bar{\mu})$.*

Proof of Corollary 2. For $\mu \in \left(\bar{\beta}, \hat{\mu}\right)$, implicitly differentiate $R(\varepsilon^*) = 0$ (with R given by (136)), and for $\mu \in (\hat{\mu}, \bar{\mu})$, implicitly differentiate $T(\varepsilon^*) = 0$ (with T given by (128)) to obtain

$$\frac{\partial \varepsilon^*}{\partial \mu} = \begin{cases} -\frac{\varepsilon^*}{\bar{\beta}\delta\theta\left(1-\bar{\beta}\pi\right)[1-G(\varepsilon^*)]+\mu-\bar{\beta}} & \text{if } \bar{\beta} < \mu < \hat{\mu} \\ -\frac{\bar{\beta}\delta\theta \int_{\varepsilon^*}^{\varepsilon_H} [1-G(\varepsilon)]d\varepsilon}{\left\{1+\bar{\beta}\delta\theta\left[\frac{\pi G(\varepsilon^*)}{1-\bar{\beta}\pi}+\frac{1-G(\varepsilon^*)}{\mu-\bar{\beta}}\right]\right\}\left(\mu-\bar{\beta}\right)^2} & \text{if } \hat{\mu} < \mu < \bar{\mu}. \end{cases}$$

Clearly, $\partial\varepsilon^*/\partial\mu < 0$ for $\mu \in (\bar{\beta}, \hat{\mu})$, and for $\mu \in (\hat{\mu}, \bar{\mu})$. ■

Proof of Proposition 4. With $\delta = 0$, in any stationary equilibrium the Euler equations for a dealer d obtained in Lemma 5 reduce to

$$\mu \geq \bar{\beta}, \text{ with " = " if } \tilde{a}_{t+1d}^m > 0$$
$$\phi^s \geq \bar{\beta}\pi\phi^s, \text{ with " = " if } \tilde{a}_{t+1d}^m > 0.$$

The maintained assumptions $\mu > \bar{\beta}$ and $\bar{\beta}\pi < 1$, and the fact that the equity will be valued in any equilibrium imply $\tilde{a}_{t+1d}^m = \tilde{a}_{t+1d}^m = 0$ for all $d \in \mathcal{D}$. Since dealers are inactive in any stationary equilibrium, we focus on investors for the remainder of the proof. In an equilibrium with no money (or no valued money), there is no trade in the OTC market. The first-order condition for an investor i in the time-t Walrasian market is

$$\phi_t^s \geq \beta\pi\mathbb{E}_t\left(\bar{\varepsilon}y_{t+1} + \phi_{t+1}^s\right), \text{ " = " if } \tilde{a}_{t+1i}^s > 0.$$

In a stationary equilibrium the Walrasian market for equity can only clear if $\phi_t^s = \frac{\bar{\beta}\pi}{1-\bar{\beta}\pi}\bar{\varepsilon}y_t$. This establishes parts (i) and (iii) in the statement of the proposition. In a stationary monetary equilibrium the Euler equations for an investor obtained in Lemma 5 reduce to

$$\mu = \bar{\beta}\left[1 + \alpha\eta \int_{\varepsilon^c}^{\varepsilon_H} \int_{\varepsilon_j}^{\varepsilon_H} \frac{\varepsilon_i - \varepsilon_j}{\varepsilon_j + \phi^s} dG(\varepsilon_i)\, dG(\varepsilon_j)\right] \tag{137}$$

$$\phi^s = \frac{\bar{\beta}\pi}{1-\bar{\beta}\pi}\left[\bar{\varepsilon} + \alpha(1-\eta)\int_{\varepsilon_L}^{\varepsilon^c}\int_{\varepsilon_L}^{\varepsilon_j} (\varepsilon_j - \varepsilon_i)\, dG(\varepsilon_i)\, dG(\varepsilon_j)\right] \tag{138}$$

where

$$\varepsilon^c \equiv \frac{Z}{A^s} - \phi^s. \tag{139}$$

Condition (137) can be substituted into (138) to obtain a single equation in the unknown ε^c, namely $\bar{T}(\varepsilon^c) = 0$, where $\bar{T} : [\varepsilon_L, \varepsilon_H] \to \mathbb{R}$ is defined by

$$\bar{T}(\varepsilon^c) \equiv \bar{\beta}\alpha\eta \int_{\varepsilon^c}^{\varepsilon_H}\int_{\varepsilon_j}^{\varepsilon_H} \frac{\varepsilon_i - \varepsilon_j}{\varepsilon_j + \frac{\bar{\beta}\pi}{1-\bar{\beta}\pi}\left[\bar{\varepsilon} + \alpha(1-\eta)\int_{\varepsilon_L}^{\varepsilon^c}\int_{\varepsilon_L}^{\varepsilon_j}(\varepsilon_j - \varepsilon_i)dG(\varepsilon_i)dG(\varepsilon_j)\right]} dG(\varepsilon_i)\, dG(\varepsilon_j) + \bar{\beta} - \mu.$$

Notice that $\bar{T}(\varepsilon_H) = \bar{\beta} - \mu < 0$ and

$$\bar{T}(\varepsilon_L) = \bar{\beta}\alpha\eta \int_{\varepsilon_L}^{\varepsilon_H}\int_{\varepsilon_j}^{\varepsilon_H} \frac{\varepsilon_i - \varepsilon_j}{\varepsilon_j + \frac{\bar{\beta}\pi}{1-\bar{\beta}\pi}\bar{\varepsilon}} dG(\varepsilon_i)\, dG(\varepsilon_j) + \bar{\beta} - \mu,$$

so since $\bar{T}$ is continuous, a stationary monetary equilibrium exists if $\mu < \tilde{\mu}$ with $\tilde{\mu}$ defined as in (35). In addition,

$$\bar{T}'(\varepsilon^c) = -\left[\bar{\beta}\alpha\eta \int_{\varepsilon^c}^{\varepsilon_H} \frac{\varepsilon_i - \varepsilon^c}{\varepsilon^c + \phi^s} dG(\varepsilon_i)\, G'(\varepsilon^c) + \frac{(\bar{\beta}\alpha)^2 \pi\eta(1-\eta)}{1-\bar{\beta}\pi}\int_{\varepsilon^c}^{\varepsilon_H}\int_{\varepsilon_j}^{\varepsilon_H} \frac{(\varepsilon_i - \varepsilon_j)\int_{\varepsilon_L}^{\varepsilon^c}(\varepsilon^c - \varepsilon)dG(\varepsilon)G'(\varepsilon^c)}{(\varepsilon_j + \phi^s)^2} dG(\varepsilon_i)\, dG(\varepsilon_j)\right]$$

is negative, so a stationary monetary equilibrium exists if and only if $\mu < \tilde{\mu}$, and there cannot be more than one stationary monetary equilibrium. Condition (36) is just (138), condition (38) is $\bar{T}(\varepsilon^c) = 0$, and (37) follows from (139). This establishes parts (ii) and (iv). Part (v) is immediate from (38). ■

Proof of Proposition 5. Recall that $\partial\varepsilon^*/\partial\mu < 0$ (Corollary 2). (i) From (27),

$$\frac{\partial\phi^s}{\partial\mu} = \frac{\bar{\beta}\pi}{1-\bar{\beta}\pi}\left[\mathbb{I}_{\{\bar{\beta}<\mu\leq\hat{\mu}\}} + \mathbb{I}_{\{\hat{\mu}<\mu<\bar{\mu}\}}\delta\theta G(\varepsilon^*)\right]\frac{\partial\varepsilon^*}{\partial\mu} < 0.$$

(ii) Condition (28) implies $\partial\bar{\phi}^s/\partial\mu = \partial\varepsilon^*/\partial\mu + \partial\phi^s/\partial\mu < 0$. (iii) Differentiate (31) to obtain

$$\frac{\partial Z}{\partial\varepsilon^*} = \delta Z \frac{G'(\varepsilon^*)A_I^s + \left[G'(\varepsilon^*)(\varepsilon^*+\phi^s)+\theta[1-G(\varepsilon^*)]\left(1+\frac{\partial\phi^s}{\partial\varepsilon^*}\right)+(1-\theta)\frac{\partial\phi^s}{\partial\varepsilon^*}\int_{\varepsilon^*}^{\varepsilon_H}\left(\frac{\varepsilon^*+\phi^s}{\varepsilon+\phi^s}\right)^2 dG(\varepsilon)\right]\frac{Z}{(\varepsilon^*+\phi^s)^2}}{A_D^s+\delta G(\varepsilon^*)A_I^s} > 0. \tag{140}$$

Hence $\partial Z/\partial\mu = (\partial Z/\partial\varepsilon^*)(\partial\varepsilon^*/\partial\mu) < 0$. From (29), $\partial\phi_t^m/\partial\mu = (y_t/A_t^m)\,\partial Z/\partial\mu < 0$. ■

Proof of Proposition 6. First, notice that $\partial\varepsilon^c/\partial\mu = 1/\bar{T}'(\varepsilon^c) < 0$, where $\bar{T}(\cdot)$ is the mapping defined in the proof of Proposition 4. (i) Differentiate (36) to obtain

$$\frac{\partial\phi^s}{\partial\mu} = \frac{\bar{\beta}\pi}{1-\bar{\beta}\pi}\alpha(1-\eta)G'(\varepsilon^c)\int_{\varepsilon_L}^{\varepsilon^c}(\varepsilon^c-\varepsilon_i)\,dG(\varepsilon_i)\frac{\partial\varepsilon^c}{\partial\mu} < 0.$$

(ii) From (37), $\partial Z/\partial\mu = (\partial\varepsilon^c/\partial\mu + \partial\phi^s/\partial\mu)A^s < 0$, and since $Z = \phi_t^m A_t^m/y_t$, $\partial\phi_t^m/\partial\mu = (\partial Z/\partial\mu)(y_t/A_t^m) < 0$. ■

Proof of Proposition 7. From condition (32),

$$\frac{\partial\varepsilon^*}{\partial(\delta\theta)} = \frac{\frac{\mu-\bar{\beta}}{\delta\theta}\left[\varepsilon^*+\bar{\beta}\pi(\bar{\varepsilon}-\varepsilon^*)\mathbb{I}_{\{\hat{\mu}<\mu\}}\right]}{\bar{\beta}\delta\theta(1-\bar{\beta}\pi)[1-G(\varepsilon^*)]+(\mu-\bar{\beta})\left\{1+\bar{\beta}\pi[\delta\theta G(\varepsilon^*)-1]\mathbb{I}_{\{\hat{\mu}<\mu\}}\right\}} > 0. \tag{141}$$

(i) From (36),

$$\frac{\partial\phi^s}{\partial(\delta\theta)} = \begin{cases} \frac{\bar{\beta}\pi}{1-\bar{\beta}\pi}\frac{\partial\varepsilon^*}{\partial(\delta\theta)} > 0 & \text{if } \bar{\beta} < \mu \le \hat{\mu} \\ \frac{\bar{\beta}\pi}{1-\bar{\beta}\pi}\left[\int_{\varepsilon_L}^{\varepsilon^*}G(\varepsilon)\,d\varepsilon + \delta\theta G(\varepsilon^*)\frac{\partial\varepsilon^*}{\partial(\delta\theta)}\right] > 0 & \text{if } \hat{\mu} < \mu < \bar{\mu}. \end{cases}$$

(ii) From (28), $\partial\bar{\phi}^s/\partial(\delta\theta) = \partial\varepsilon^*/\partial(\delta\theta) + \partial\phi^s/\partial(\delta\theta) > 0$. (iii) For $\mu \in (\hat{\mu},\bar{\mu})$, (31) implies $\partial Z/\partial\delta = (\partial Z/\partial\varepsilon^*)(\partial\varepsilon^*/\partial\delta) > 0$ (the sign follows from (140) and (141)), and therefore $\partial\phi_t^m/\partial\delta = (\partial Z/\partial\delta)(y_t/A_t^m) > 0$. ■

Proof of Proposition 8. Implicit differentiation of $\bar{T}(\varepsilon^c) = 0$ implies

$$\frac{\partial\varepsilon^c}{\partial\alpha} = \frac{\int_{\varepsilon^c}^{\varepsilon_H}\int_{\varepsilon_j}^{\varepsilon_H}\frac{\eta(1-\bar{\beta}\pi)(\varepsilon_i-\varepsilon_j)\left[(1-\bar{\beta}\pi)\varepsilon_j+\bar{\beta}\pi\bar{\varepsilon}\right]}{\left\{(1-\bar{\beta}\pi)\varepsilon_j+\bar{\beta}\pi[\bar{\varepsilon}+\alpha(1-\eta)\varphi(\varepsilon^c)]\right\}^2}dG(\varepsilon_i)dG(\varepsilon_j)}{\int_{\varepsilon^c}^{\varepsilon_H}\frac{\alpha\eta(1-\bar{\beta}\pi)(\varepsilon_i-\varepsilon_j)}{(1-\bar{\beta}\pi)\varepsilon^c+\bar{\beta}\pi[\bar{\varepsilon}+\alpha(1-\eta)\varphi(\varepsilon^c)]}dG(\varepsilon_i)G'(\varepsilon^c)+\int_{\varepsilon^c}^{\varepsilon_H}\int_{\varepsilon_j}^{\varepsilon_H}\frac{\bar{\beta}\pi\alpha^2\eta(1-\eta)(1-\bar{\beta}\pi)(\varepsilon_i-\varepsilon_j)\varphi'(\varepsilon^c)}{\left\{(1-\bar{\beta}\pi)\varepsilon_j+\bar{\beta}\pi[\bar{\varepsilon}+\alpha(1-\eta)\varphi(\varepsilon^c)]\right\}^2}dG(\varepsilon_i)dG(\varepsilon_j)} > 0.$$

(i) Differentiate (36) to arrive at

$$\frac{\partial\phi^s}{\partial\alpha} = \frac{\bar{\beta}\pi(1-\eta)}{1-\bar{\beta}\pi}\left[\varphi(\varepsilon^c)+\alpha\int_{\varepsilon_L}^{\varepsilon^c}(\varepsilon^c-\varepsilon_i)\,dG(\varepsilon_i)\,dG(\varepsilon^c)\frac{\partial\varepsilon^c}{\partial\alpha}\right] > 0.$$

(ii) From (37),

$$\frac{\partial Z}{\partial \alpha} = \left(\frac{\partial \varepsilon^c}{\partial \alpha} + \frac{\partial \phi^s}{\partial \alpha} \right) A^s > 0,$$

and since $Z = \phi_t^m A_t^m / y_t$, it follows that $\partial \phi_t^m / \partial \alpha > 0$. ■

Proof of Proposition 9. (i) The result is immediate from the expression for A_D^s in Proposition 3. (ii) From (24) and (25),

$$\frac{\partial \hat{\mu}}{\partial (\delta\theta)} = \bar{\beta} \left(1 - \bar{\beta}\pi \right) \left\{ \frac{(1-\delta\theta)\,\bar{\varepsilon}}{\left[1 - \delta\theta G(\hat{\varepsilon})\right] \hat{\varepsilon}^2} \int_{\varepsilon_L}^{\hat{\varepsilon}} G(\varepsilon)\, d\varepsilon - \frac{\hat{\varepsilon} - \bar{\varepsilon}}{\hat{\varepsilon}} \right\}.$$

Notice that $\partial \hat{\mu} / \partial (\delta\theta)$ approaches a positive value as $\delta\theta \to 0$, and a negative value as $\delta\theta \to 1$. Also, $\hat{\mu} \to \bar{\beta}$ both when $\delta\theta \to 0$, and when $\delta\theta \to 1$. Hence $\mu > \bar{\beta} = \lim_{\delta\theta \to 0} \hat{\mu} = \lim_{\delta\theta \to 1} \hat{\mu}$ for a range of values of $\delta\theta$ close to 0 and a range of values of $\delta\theta$ close to 1. For those ranges of values of $\delta\theta$, $A_D^s = 0$. In between those ranges there must exist values of $\delta\theta$ such that $\mu < \hat{\mu}$ which implies $A_D^s > 0$. ■

Proof of Proposition 10. (i) Differentiate (39) to get

$$\frac{\partial \mathcal{V}}{\partial \mu} = 2\delta G'(\varepsilon^*) (A^s - \pi \tilde{A}_D^s) \frac{\partial \varepsilon^*}{\partial \mu} < 0,$$

where the inequality follows from Corollary 2. (ii) From (39),

$$\frac{\partial \mathcal{V}}{\partial \theta} = 2\delta G'(\varepsilon^*) (A^s - \pi \tilde{A}_D^s) \frac{\partial \varepsilon^*}{\partial \theta}$$
$$\frac{\partial \mathcal{V}}{\partial \delta} = 2 \left[G(\varepsilon^*) + \delta G'(\varepsilon^*) \frac{\partial \varepsilon^*}{\partial \delta} \right] (A^s - \pi \tilde{A}_D^s)$$

and both are positive since $\partial \varepsilon^* / \partial (\delta\theta) > 0$ (see (141)). ■

Proof of Proposition 10. Rewrite $\tilde{\mathcal{V}}$ as

$$\tilde{\mathcal{V}} = \alpha A^s \int_{\varepsilon_L}^{\varepsilon^c} \left\{ \eta \left[1 - G(\varepsilon_i)\right] + (1-\eta) G(\varepsilon_i) \right\} dG(\varepsilon_i)$$
$$+ \alpha A^s \int_{\varepsilon^c}^{\varepsilon_H} \left\{ \eta \left[1 - G(\varepsilon_i)\right] + (1-\eta) G(\varepsilon_i) \right\} \frac{\varepsilon^c + \phi^s}{\varepsilon_i + \phi^s} dG(\varepsilon_i).$$

Differentiate to obtain

$$\frac{\partial \tilde{\mathcal{V}}}{\partial \varepsilon^c} = \alpha A^s \int_{\varepsilon^c}^{\varepsilon_H} \left\{ \eta \left[1 - G(\varepsilon_i)\right] + (1-\eta) G(\varepsilon_i) \right\} \frac{\partial}{\partial \varepsilon^c} \left[\frac{\varepsilon^c + \phi^s}{\varepsilon_i + \phi^s} \right] dG(\varepsilon_i),$$

where

$$\frac{\partial}{\partial \varepsilon^c}\left[\frac{\varepsilon^c+\phi^s}{\varepsilon_i+\phi^s}\right]=\frac{\varepsilon_i+\phi^s+(\varepsilon_i-\varepsilon^c)\frac{\partial \phi^s}{\partial \varepsilon^c}}{(\varepsilon_i+\phi^s)^2}A^s>0 \text{ for } \varepsilon_i>\varepsilon^c.$$

Hence, $\partial\tilde{\mathcal{V}}/\partial\varepsilon^c>0$. Thus $\partial\tilde{\mathcal{V}}/\partial\mu=(\partial\tilde{\mathcal{V}}/\partial\varepsilon^c)(\partial\varepsilon^c/\partial\mu)<0$, since $\partial\varepsilon^c/\partial\mu<0$ (see proof of Proposition 6), which establishes (i). For part (ii), simply notice that $\partial\tilde{\mathcal{V}}/\partial\alpha=\tilde{\mathcal{V}}/\alpha+(\partial\tilde{\mathcal{V}}/\partial\varepsilon^c)(\partial\varepsilon^c/\partial\alpha)>0$. ■

Proof of Proposition 12. (i) For $\bar{\beta}<\mu\leq\hat{\mu}$, $\partial\mathcal{P}/\partial\mu=\left[\bar{\beta}\pi/(1-\bar{\beta}\pi)\right](\partial\varepsilon^*/\partial\mu)<0$, and for $\hat{\mu}<\mu<\bar{\mu}$, $\partial\mathcal{P}/\partial\mu=\left[\bar{\beta}\pi/(1-\bar{\beta}\pi)\right]\delta\theta G(\varepsilon^*)(\partial\varepsilon^*/\partial\mu)<0$. (ii) For $\bar{\beta}<\mu\leq\hat{\mu}$, $\partial\mathcal{P}/\partial(\delta\theta)=\left[\bar{\beta}\pi/(1-\bar{\beta}\pi)\right](\partial\varepsilon^*/\partial(\delta\theta))>0$, and for $\hat{\mu}<\mu<\bar{\mu}$, $\partial\mathcal{P}/\partial\mu=\left[\bar{\beta}\pi/(1-\bar{\beta}\pi)\right]\{\delta\theta G(\varepsilon^*)[\partial\varepsilon^*/\partial(\delta\theta)]+\int_{\varepsilon_L}^{\varepsilon^*}G(\varepsilon)\,d\varepsilon\}>0$. ■

Proof of Proposition 13. (i) $\partial\tilde{\mathcal{P}}/\partial\mu=\left[\bar{\beta}\pi/(1-\bar{\beta}\pi)\right]\alpha(1-\eta)\varphi'(\varepsilon^c)(\partial\varepsilon^c/\partial\mu)<0$. (ii) $\partial\tilde{\mathcal{P}}/\partial\alpha=\left[\bar{\beta}\pi/(1-\bar{\beta}\pi)\right](1-\eta)\{\alpha\varphi'(\varepsilon^c)(\partial\varepsilon^c/\partial\alpha)+\varphi(\varepsilon^c)\}>0$. ■

Proof of Proposition 14. The choice variable a'_{tD} does not appear in the Planner's objective function, so $a'_{tD}=0$ at an optimum. Since (42) must bind for every t at an optimum, the planner's problem is equivalent to

$$W^{**}=\max_{\left\{v_t,\tilde{a}_{tD},\tilde{a}_{tI},a'_{tI}\right\}_{t=0}^{\infty}}\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\left\{\delta(v_t)\int_{[\varepsilon_L,\varepsilon_H]}\varepsilon a'_{tI}(d\varepsilon)+[1-\delta(v_t)]\bar{\varepsilon}a_{tI}-kv_{t+1}\right\}y_t$$

subject to (6), (7), (40) and (41). Clearly, $\int_{[\varepsilon_L,\varepsilon_H]}\varepsilon a'_{tI}(d\varepsilon)\leq\varepsilon_H$ and (41) must bind at an optimum, so $W^{**}\leq\bar{W}^{**}$, where

$$\begin{aligned}\bar{W}^{**}&=\max_{\{v_t,\tilde{a}_{tD},\tilde{a}_{tI}\}_{t=0}^{\infty}}\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\left\{[v_ta_{tD}+\delta(v_t)a_{tI}]\varepsilon_H+[1-\delta(v_t)]\bar{\varepsilon}a_{tI}-kv_{t+1}\right\}y_t\\&=\max_{\{v_t,\tilde{a}_{tI}\}_{t=0}^{\infty}}\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\left\{[\pi\varepsilon_H+(1-\pi)\{\delta(v_t)\varepsilon_H+[1-\delta(v_t)]\bar{\varepsilon}\}]A^s\right.\\&\left.-[1-\delta(v_t)](\varepsilon_H-\bar{\varepsilon})\pi\tilde{a}_{tI}-kv_{t+1}\right\}y_t\\&=\max_{\{v_t\}_{t=0}^{\infty}}\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\left\{[\pi\varepsilon_H+(1-\pi)\{\delta(v_t)\varepsilon_H+[1-\delta(v_t)]\bar{\varepsilon}\}]A^s-kv_{t+1}\right\}y_t\\&=\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\left\{[\pi\varepsilon_H+(1-\pi)\{\delta(v_t^*)\varepsilon_H+[1-\delta(v_t^*)]\bar{\varepsilon}\}]A^s-kv_{t+1}^*\right\}y_t,\end{aligned}$$

where the maximization in the first line is subject to (6), (7) and (40) (which must bind at an optimum), the second line has been obtained by substituting these constraints into the objective function, and $\{v_t^*\}$ in the last line denotes the sequence of v_t characterized by (43). The allocation in the statement of the proposition achieves $\bar{W}^{**}$ and therefore solves the Planner's problem. ■

Lemma 8 *In any equilibrium, the free-entry condition (44) can be written as (45).*

Proof of Lemma 8. With (14), the left side of condition (44) can be written as

$$\max_{\left(a_{t+1}^m, a_{t+1}^s\right) \in \mathbb{R}_+^2} \left[\beta \mathbb{E}_t V_{t+1}^D\left(a_{t+1}^m, \pi a_{t+1}^s\right) - \left(\phi_t^m a_{t+1}^m + \phi_t^s a_{t+1}^s\right)\right] - k_t.$$

And with (17), this last expression becomes

$$\max_{\left(a_{t+1}^m, a_{t+1}^s\right) \in \mathbb{R}_+^2} \left[\left(\beta \mathbb{E}_t \bar{\phi}_{t+1} - \phi_t^m\right) a_{t+1}^m + \left(\beta \pi \mathbb{E}_t \bar{\phi}_{t+1} p_{t+1} - \phi_t^s\right) a_{t+1}^s\right] + \beta \mathbb{E}_t V_{t+1}^D(\mathbf{0}) - k_t, \quad (142)$$

where

$$V_{t+1}^D(\mathbf{0}) \equiv \kappa\left(v_{t+1}\right)(1-\theta) \bar{\phi}_{t+1}\left[A_{It+1}^m \int_{\varepsilon_{t+1}^*}^{\varepsilon_H} \frac{\left(\varepsilon-\varepsilon_{t+1}^*\right) y_{t+1}}{\varepsilon y_{t+1}+\phi_{t+1}^s} dG(\varepsilon) + p_{t+1} A_{It+1}^s \int_{\varepsilon_L}^{\varepsilon_{t+1}^*} \frac{\left(\varepsilon_{t+1}^*-\varepsilon\right) y_{t+1}}{\varepsilon_{t+1}^* y_{t+1}+\phi_{t+1}^s} dG(\varepsilon)\right]$$
$$+ \max\left\{W_{t+1}^D(\mathbf{0}) - k_t, 0\right\}$$

is as in Lemma 4, except for the last term, which reflects the fact that the dealer has to bear cost k in order to participate in the OTC market of the following period. In equilibrium, the dealer optimization (conditions (19) and (20)) implies

$$\max_{\left(a_{t+1}^m, a_{t+1}^s\right) \in \mathbb{R}_+^2} \left[\left(\beta \mathbb{E}_t \bar{\phi}_{t+1} - \phi_t^m\right) a_{t+1}^m + \left(\beta \pi \mathbb{E}_t \bar{\phi}_{t+1} p_{t+1} - \phi_t^s\right) a_{t+1}^s\right] = 0.$$

Also, (44) implies $\max\left\{W_{t+1}^D(\mathbf{0}) - k_t, 0\right\} = 0$. Hence (142) reduces to $\Phi_{t+1} - k_t$, with Φ_{t+1} as defined below (45). ■

Proof of Proposition 15. Consider a stationary equilibrium with free entry (for the model with $\alpha = 0$). As $\mu \to \bar{\beta}$, (32) implies

$$\frac{\int_{\varepsilon^*}^{\varepsilon_H}[1-G(\varepsilon)]\, d\varepsilon}{\varepsilon^* + \beta\pi\left[\bar{\varepsilon} - \varepsilon^* + \delta(v)\,\theta \int_{\varepsilon_L}^{\varepsilon^*} G(\varepsilon)\, d\varepsilon\right] \mathbb{I}_{\{\hat{\mu}<\mu\}}} \to 0$$

which in turn implies $\varepsilon^* \to \varepsilon_H$. The dealer's and the investor's Euler equations for equity in Lemma 5 imply

$$\phi^s = \frac{\bar{\beta}\pi}{1-\bar{\beta}\pi} \max\left\{\varepsilon^*, \bar{\varepsilon} + \delta(v)\,\theta \int_{\varepsilon_L}^{\varepsilon^*} (\varepsilon^* - \varepsilon) dG(\varepsilon)\right\},$$

and as $\varepsilon^* \to \varepsilon_H$, $\max\left\{\varepsilon^*, \bar{\varepsilon} + \delta(v)\,\theta \int_{\varepsilon_L}^{\varepsilon^*} (\varepsilon^* - \varepsilon) dG(\varepsilon)\right\} \to \max\{\varepsilon_H, \bar{\varepsilon} + \delta(v)\,\theta(\varepsilon_H - \bar{\varepsilon})\} = \varepsilon_H$, so $\tilde{A}_D^s \to A^s$, i.e., only dealers hold equity overnight. Thus, from (48), $\bar{\Phi} - k \to \Pi(v)$, where

$$\Pi(v) \equiv \bar{\beta}\kappa(v)(1-\theta)(\varepsilon_H - \bar{\varepsilon})(1-\pi)A^s - k.$$

Notice that

$$\lim_{v\to\infty} \Pi(v) = -k < 0 < \bar{\beta}(1-\theta)(\varepsilon_H - \bar{\varepsilon})(1-\pi)A^s - k = \Pi(0)$$

and $\Pi'(v) = \bar{\beta}\kappa'(v)(1-\theta)(\varepsilon_H - \bar{\varepsilon})(1-\pi)A^s < 0$, so there exists a unique $v \in (0, \infty)$ that satisfies $\Pi(v) = 0$. To conclude, we only need to show that under the hypothesis of the proposition, $\Pi(v) = 0$ is equivalent to (43). Notice that $\delta''(v) < 0$ implies $\kappa(v) = \delta(v)/v \leq \delta'(0)$ for any $v \geq 0$. In particular, for $v = 0$ this implies $1 \leq \delta'(0)$. Hence

$$0 < \bar{\beta}(1-\theta)(\varepsilon_H - \bar{\varepsilon})(1-\pi)A^s - k \leq \bar{\beta}\delta'(0)(\varepsilon_H - \bar{\varepsilon})(1-\pi)A^s - k$$

which means that $v > 0$ in the Planner's solution. Then (43) must hold with equality and the optimal v satisfies

$$\bar{\beta}\delta'(v)(\varepsilon_H - \bar{\varepsilon})(1-\pi)A^s - k = 0. \tag{143}$$

Finally, notice that $\delta'(v) = \kappa(v) + \kappa'(v)\,v$, so if $1-\theta = 1 - \frac{-\kappa'(v)v}{\kappa(v)} = \frac{\delta'(v)}{\kappa(v)}$ then (143) is identical to $\Pi(v) = 0$. ■