Online Appendix for "The Informativeness Principle Under Limited Liability"

Pierre Chaigneau, Alex Edmans, and Daniel Gottlieb

B Continuous Effort Model

The agent's effort is e, where $e \in [0, \infty)$. Conditional on effort e and signal s, output q is continuously distributed on $[0, \overline{q}]$ according to the density $f(q|e,s)$, which satisfies MLRP: $\frac{d}{dq}\frac{f'_e(q|e,s)}{f(q|e,s)} > 0$, where f'_e denotes the partial derivative of f with respect to e.

Let $s \in \{s_1, \ldots, s_S\}$ be another signal of effort, with $\phi_{\hat{e}}^{\hat{s}} = Pr(s = \hat{s}|e = \hat{e}) \in (0,1)$. Let $f(q|e) = \sum_s \phi_e^s f(q|e,s)$. For technical reasons, assume that f'_q, f'_e, and f''_{qe} exist everywhere, where f'_q denotes the partial derivative of f with respect to q, and f''_{qe} denotes the cross-partial derivative of f with respect to q and e. The agent's cost of effort is $C(e)$, where $C' > 0$ and $C'' > 0$. We define his expected utility as:

$$U(w,e) = \sum_s \phi_e^s \int_0^{\overline{q}} w(q,s) f(q|e,s) dq - C(e) \tag{34}$$

B.1 Bilateral Limited Liability

As in Section 2.1, we assume limited liability for both principal and agent ($0 \leq w(q,s) \leq q$, $\forall \{q,s\}$). These constraints imply that $w(q,s)$ is continuous and differentiable with respect to q almost everywhere.

As in the first stage of Grossman and Hart (1983), assume that the principal wishes to implement a target effort level $\widehat{e}$; our results below will hold for any given $\widehat{e}$, including the (here undetermined) optimal level of $\widehat{e}$.[7]

[7] The *existence* of the signal may affect the optimal level of effort. However, even if it does, the *realization* of the signal may or may not affect the payment to the agent, which is the question that we study in this section.

For a given e, the principal's problem is:

$$\max_{w} \sum_{s} \phi_{\widehat{e}}^{s} \int_{0}^{\overline{q}} \left[q - w(q,s)\right] f(q|\widehat{e}, s) dq \tag{35}$$

$$\text{s.t.} \qquad \widehat{e} \in \arg\max_{e} U(w,e) \tag{36}$$

$$U(w, \widehat{e}) \geq 0 \tag{37}$$

$$w(q,s) \in [0,q], \forall \{q,s\}. \tag{38}$$

Assume that the cost of effort function is sufficiently convex for the second-order condition to the effort choice problem to be satisfied, so that the first-order approach is valid. Formally, we impose

$$\sup_{\{z_s\}} \sum_{s} \left\{ \frac{d^2\phi_e^s}{de^2} \int_{z_s}^{\overline{q}} q f(q|e,s) dq + 2 \frac{d\phi_e^s}{de} \int_{z_s}^{\overline{q}} q f_e'(q|e,s) dq + \phi_e^s \int_{z_s}^{\overline{q}} q f_{ee}''(q|e,s) dq \right\} < C''(e) \ \forall e \tag{39}$$

where f_{ee}'' is the second-order partial derivative of f with respect to e. Note that (36) implies that $U(w,e) \geq U(w,0)$. Due to LL, we have $U(w,0) \geq 0$ and so (36) implies (37).

Lemma 5 *To induce effort level $\widehat{e}$, the optimal contract is*

$$w(q,s) = \begin{cases} 0 & \text{if } q < z_s(\widehat{e}), \\ q & \text{if } q \geq z_s(\widehat{e}). \end{cases} \tag{40}$$

for thresholds $z_s(\widehat{e})$ corresponding to the different realizations of the signal.

Proof. Since the first-order approach is valid, and given a contract $w(q,s)$, IC (36) may be rewritten as

$$\sum_{s} \left[\frac{d\phi_{\widehat{e}}^s}{de} \int_{0}^{\overline{q}} w(q,s) f(q|\widehat{e},s) dq + \phi_{\widehat{e}}^s \int_{0}^{\overline{q}} w(q,s) f_e'(q|\widehat{e},s) dq \right] = C'(\widehat{e}) \tag{41}$$

The principal's objective function is linear in $w(q,s)$. Given two-sided limited liability, the solution is given by

$$w(q,s) = \begin{cases} 0 & \text{if } A_s(q) < 0, \\ q & \text{if } A_s(q) > 0. \end{cases} \tag{42}$$

where

$$A_s(q) \equiv -\phi^s_{\widehat{e}} f(q|\widehat{e},s) + \lambda \left[\frac{d\phi^s_{\widehat{e}}}{de} f(q|\widehat{e},s) + \phi^s_{\widehat{e}} f'_e(q|\widehat{e},s) \right] \quad (43)$$

where λ denotes the Lagrange multiplier associated with (41). Then

$$A_s(q) > 0 \iff \frac{d\phi^s_{\widehat{e}}/de}{\phi^s_{\widehat{e}}} + \frac{f'_e(q|\widehat{e},s)}{f(q|\widehat{e},s)} > \frac{1}{\lambda} \quad (44)$$

Due to MLRP, (44) is satisfied for a given s if and only if q exceeds a threshold, which we denote by $z_s(\widehat{e})$. ■

Proposition 4 gives a condition under which the signal has zero value for the contract.

Proposition 4 *Given effort $\widehat{e}$ to be induced, if $\forall$ s, $\frac{d\phi^s_{\widehat{e}}}{de} = 0$ and $\frac{f'_e(z_s|\widehat{e},s)}{f(z_s|\widehat{e},s)}$ does not depend on s at $z_s = z^*$, where z^* is the maximum threshold independent of s that solves (41), then $z_s = z^*$ $\forall$ s.*

Proof. If $\frac{d\phi^s_{\widehat{e}}}{de} = 0$ and given the optimal contract in Lemma 5, the IC in (41) may be rewritten as

$$\sum_s \phi^s_{\widehat{e}} \int_{z_s}^{\overline{q}} q f'_e(q|\widehat{e},s) dq = C'(\widehat{e}). \quad (45)$$

Let z^* be the highest threshold independent of the signal realization s that solves (45).[8] Now, suppose that $\frac{d\phi^s_{\widehat{e}}}{de} = 0$ and that $\frac{f'_e(z_s|\widehat{e},s)}{f(z_s|\widehat{e},s)}$ does not depend on s at $z_s = z^*$. Then, according to (42) and (44), the threshold $z_s(\widehat{e})$ does not depend on s, and it is equal to z^* $\forall$ s. ■

This result means that if the signal s is not informative about marginal changes of effort from the implemented effort level (i.e., if $d\phi^s_{\widehat{e}}/de = 0$), and if the likelihood ratio of output $\frac{f'_e(q|\widehat{e},s)}{f(q|\widehat{e},s)}$ does not depend on the signal s for $q = z^*$, then the wage does not depend on the realization of the signal s. Note that these conditions can be satisfied even if $d\phi^s_e/de > 0$ almost everywhere, and even if the likelihood ratio of output depends on the signal almost everywhere.

[8]If $\frac{d\phi^s_{\widehat{e}}}{de} = 0$, and if there are several thresholds independent of the signal realization that solve (45) and therefore elicit effort $\widehat{e}$, cost minimization imposes that the principal chooses the highest threshold.

B.2 Monotonicity Constraint

We now also impose a monotonicity constraint as in Innes (1990): $\forall\ \{q, s\}$,

$$0 \leq w\,(q+\epsilon) - w\,(q) \leq \epsilon\ \forall\ \epsilon. \tag{46}$$

For simplicity, we assume that the likelihood ratio is unbounded from above:

$$\lim_{q \to \overline{q}} \frac{f'_e(q|\widehat{e}, s)}{f(q|\widehat{e}, s)} = \infty \tag{47}$$

$\forall\ s$. As in the baseline model, this assumption allows to rule out corner solutions, thereby ensuring that the strike price is lower than $\overline{q}$ for all signal realizations.

For a given e, the principal's problem is given by (35)-(38), with the additional monotonicity constraint (46). Assume that the cost of effort function is sufficiently convex for the second-order condition to the effort choice problem to be satisfied, so that the first-order approach is valid. Formally, we impose

$$\sup_{\{z_s\}} \sum_s \left\{ \frac{d^2\phi_e^s}{de^2} \int_{z_s}^{\overline{q}} (q - z_s) f(q|e, s) dq + 2\frac{d\phi_e^s}{de} \int_{z_s}^{\overline{q}} (q - z_s) f'_e(q|e, s) dq + \phi_e^s \int_{z_s}^{\overline{q}} (q - z_s) f''_{ee}(q|e, s) dq \right\} < C''(e)\ \forall e$$

As in the previous section, (36) implies (37). Let $q = q(\theta, e)$, where θ is a random variable with PDF g and CDF G which is independent of e; we assume that $q = q(\theta, e)$ is twice differentiable with respect to θ and e, and that $q'_\theta(\theta, e) > 0$ and $q'_e(\theta, e) > 0$, where $q'_\theta(\theta, e)$ and $q'_e(\theta, e)$ are the partial derivatives of q with respect to θ and e, respectively. Then Lemma 6 gives the optimal contract and Proposition 5 gives a condition under which the contract does not depend on the signal.

Lemma 6 *Conditional on s, the optimal contract is $w(q, s) = \max\{q - z_s, 0\}$ for some z_s.*

Proof. We prove this Lemma by contradiction, as in Poblete and Spulber (2012). For any given realization s of the signal, let the "critical ratio" be defined as

$$\rho(\theta, e) \equiv \frac{g(\theta)}{1 - G(\theta)} \frac{q'_e(\theta, e)}{q'_\theta(\theta, e)}.$$

According to Proposition 3 in Poblete and Spulber (2012), MLRP implies that the critical ratio is increasing in θ. It remains to be shown that for any given s_i, and holding constant the contract for other signal realizations, any contract which is not of the form $w(q,s) = \max\{q - z_s, 0\}$ for some z_s is dominated. The Proof follows the same lines as the proof of Proposition 1 in Poblete and Spulber (2012). ■

Proposition 5 *Given effort* $\widehat{e}$ *to be induced, if* $d\phi^s_{\widehat{e}}/de = 0$ *and* $\frac{\int_{z_s}^{\overline{q}} f'_e(q|\widehat{e},s)dq}{\int_{z_s}^{\overline{q}} f(q|\widehat{e},s)dq}$ *does not depend on* s *at* $z_s = z^{**}$*, where* z^{**} *is determined by (54), then* $z_s = z^{**}\ \forall\ s$*.*

Proof. For a given level of effort $\widehat{e}$ to be induced, and given Lemma 6, the principal's problem is

$$\min_{\{z_s\}} \sum_s \phi^s_{\widehat{e}} \int_{z_s}^{\overline{q}} (q - z_s) f(q|\widehat{e},s)dq \tag{48}$$

$$\text{s.t.} \quad \sum_s \left[\frac{d\phi^s_{\widehat{e}}}{de} \int_{z_s}^{\overline{q}} (q - z_s) f(q|\widehat{e},s)dq + \phi^s_{\widehat{e}} \int_{z_s}^{\overline{q}} (q - z_s) f'_e(q|\widehat{e},s)dq \right] = C'(\widehat{e}) \tag{49}$$

The first-order condition with respect to z_s is

$$-\phi^s_{\widehat{e}} \int_{z_s}^{\overline{q}} f(q|\widehat{e},s)dq - \lambda \left(\frac{d\phi^s_{\widehat{e}}}{de} \int_{z_s}^{\overline{q}} -f(q|\widehat{e},s)dq + \phi^s_{\widehat{e}} \int_{z_s}^{\overline{q}} -f'_e(q|\widehat{e},s)dq \right) = 0 \tag{50}$$

where $\lambda > 0$ is the Lagrange multiplier associated with the IC (49). The equation in (50) can be rewritten

$$-\phi^s_{\widehat{e}} + \lambda \left(\frac{d\phi^s_{\widehat{e}}}{de} + \phi^s_{\widehat{e}} \frac{\int_{z_s}^{\overline{q}} f'_e(q|\widehat{e},s)dq}{\int_{z_s}^{\overline{q}} f(q|\widehat{e},s)dq} \right) = 0 \tag{51}$$

where the threshold z_s in (51) is a critical point such that the derivative of the Lagrangian is zero.

For any threshold $z_s \leq 0$, and since $f(q|\widehat{e},s) = f'_e(q|\widehat{e},s) = 0$ for $q < 0$, the first derivative of the Lagrangian with respect to z_s is

$$\mathcal{L}'(z_s) = -\phi^s_{\widehat{e}} \int_0^{\overline{q}} f(q|\widehat{e},s)dq - \lambda \left(\frac{d\phi^s_{\widehat{e}}}{de} \int_0^{\overline{q}} -f(q|\widehat{e},s)dq + \phi^s_{\widehat{e}} \int_0^{\overline{q}} -f'_e(q|\widehat{e},s)dq \right) = -\phi^s_{\widehat{e}} + \lambda \frac{d\phi^s_{\widehat{e}}}{de}, \tag{52}$$

since $\int_0^{\overline{q}} f(q|\widehat{e},s)dq = 1$ and $\int_0^{\overline{q}} f'_e(q|\widehat{e},s)dq = 0$. Comparing with (**??**), and since MLRP implies $\int_{z_s}^{\overline{q}} f'_e(q|\widehat{e},s)dq > 0$, we have $\mathcal{L}'(z_s) < 0$ for $z_s \leq 0$. Therefore, the optimal z_s which solves the minimization problem in (48) must be positive for any s. Note that this ensures that the principal's LL is satisfied.

We also need to establish that $z_s < \bar{q}$ $\forall s$ to rule out corner solutions. First, note that a contract with $z_s \geq \bar{q}$ is equivalent to one with $z_s = \bar{q}$. Thus, we need to check that $\lim_{z_s \to \bar{q}^-} \mathcal{L}'(z_s) > 0$. We have

$$\lim_{z_s \to \bar{q}^-} \mathcal{L}'(z_s) = \lim_{y \to \bar{q}^-} \left\{ -\phi^s_{\widehat{e}} f(y|\widehat{e}, s)(\bar{q} - y) + \lambda \left(\frac{d\phi^s_{\widehat{e}}}{de} f(y|\widehat{e}, s) + \phi^s_{\widehat{e}} f'_e(y|\widehat{e}, s) \right) (\bar{q} - y) \right\} \tag{53}$$

where the expression in brackets has the same sign as $-\phi^s_{\widehat{e}} + \lambda \left(\frac{d\phi^s_{\widehat{e}}}{de} + \phi^s_{\widehat{e}} \frac{f'_e(y|\widehat{e},s)}{f(y|\widehat{e},s)} \right)$. Since assumption (47) implies $\lim_{y \to \bar{q}} \frac{f'_e(y|\widehat{e},s)}{f(y|\widehat{e},s)} = \infty$, we indeed have $\lim_{z_s \to \bar{q}^-} \mathcal{L}'(z_s) > 0$ if $\lambda > 0$, which follows from standard arguments.

These results ensure that, $\forall$ s, the optimal z_s is a critical point that lies in the interval $(0, \bar{q})$. It is therefore described by the necessary first-order condition in (50). If $\frac{d\phi^s_{\widehat{e}}}{de} = 0$, and given the optimal contract in Lemma 6, the IC in (49) may be rewritten as

$$\sum_s \phi^s_{\widehat{e}} \int_{z_s}^{\bar{q}} (q - z) f'_e(q|\widehat{e}, s) dq = C'(\widehat{e}) \tag{54}$$

Let z^{**} be the highest threshold independent of s that solves (54).[9] Rearranging (50), z_s does not depend on s if and only if the level of z_s that solves

$$\frac{d\phi^s_{\widehat{e}}/de}{\phi^s_{\widehat{e}}} + \frac{\int_{z_s}^{\bar{q}} f'_e(q|\widehat{e}, s) dq}{\int_{z_s}^{\bar{q}} f(q|\widehat{e}, s) dq} = \frac{1}{\lambda} \tag{55}$$

does not depend on s. Since the likelihood ratio $\frac{f'_e(\cdot|\widehat{e},s)}{f(\cdot|\widehat{e},s)}$ is strictly increasing by assumption, sufficient conditions for Z_s to be independent of s are $d\phi^s_{\widehat{e}}/de = 0$, and $\frac{\int_{z^{**}}^{\bar{q}} f'_e(q|\widehat{e},s)dq}{\int_{z^{**}}^{\bar{q}} f(q|\widehat{e},s)dq} = \frac{\int_{z^{**}}^{\bar{q}} f'_e(q|\widehat{e},s')dq}{\int_{z^{**}}^{\bar{q}} f(q|\widehat{e},s')dq}$ for any pair s, s', where z^{**} solves (54) with equality.

The second-order condition to the optimization problem in (48) are

$$\phi^s_{\widehat{e}} f(z_s|\widehat{e}, s) - \lambda \left(\frac{d\phi^s_{\widehat{e}}}{de} f(z_s|\widehat{e}, s) dq + \phi^s_{\widehat{e}} f'_e(z_s|\widehat{e}, s) dq \right) \geq 0 \tag{56}$$

[9] If $\frac{d\phi^s_{\widehat{e}}}{de} = 0$, and if there are several thresholds independent of the signal realization that solve the IC and therefore elicit effort $\widehat{e}$, cost minimization imposes that the principal chooses the highest threshold.

For $d\phi^s_{\widehat{e}}/de = 0$ and $\frac{\int_{z^{**}}^{\overline{q}} f'_e(q|\widehat{e},s)dq}{\int_{z^{**}}^{\overline{q}} f(q|\widehat{e},s)dq} = \frac{\int_{z^{**}}^{\overline{q}} f'_e(q|\widehat{e},s')dq}{\int_{z^{**}}^{\overline{q}} f(q|\widehat{e},s')dq}$ for any pair s, s', (50) gives

$$\frac{\int_{z_s}^{\overline{q}} f'_e(q|\widehat{e})dq}{\int_{z_s}^{\overline{q}} f(q|\widehat{e})dq} = \frac{1}{\lambda} \tag{57}$$

where z_s is a critical point. In this case, for a given s, (56) is satisfied if and only if

$$\frac{\int_{z_s}^{\overline{q}} f'_e(q|\widehat{e},s)dq}{\int_{z_s}^{\overline{q}} f(q|\widehat{e},s)dq} \geq \frac{f'_e(z_s|\widehat{e},s)}{f(z_s|\widehat{e},s)} \tag{58}$$

This condition is satisfied for $f'_e(q|e^*,s) \leq 0$: indeed, the RHS of (58) is then negative whereas the LHS is positive since $\int_0^{\overline{q}} f'_e(q|\widehat{e},s)dq = 0$ and that the likelihood ratio is increasing. This condition is also satisfied for $f'_e(q|\widehat{e},s) > 0$: in this case, (58) may be rewritten as

$$\int_{z_s}^{\overline{q}} \frac{f'_e(q|\widehat{e},s)}{f'_e(z_s|\widehat{e},s)}dq - \int_{z_s}^{\overline{q}} \frac{f(q|\widehat{e},s)}{f(z_s|\widehat{e},s)}dq \geq 0. \tag{59}$$

This condition is satisfied since (i) it holds as an equality for $z_s = \overline{q}$ and (ii) the LHS of (59) is strictly decreasing in z_s for $q > z_s$ given

$$\frac{f'_e(q|\widehat{e},s)}{f'_e(z_s|\widehat{e},s)} - \frac{f(q|\widehat{e},s)}{f(z_s|\widehat{e},s)} > 0 \tag{60}$$

which follows from MLRP and $q > z_s$. Thus, any critical point is a minimum. ■

This result means that if the signal s is not informative about marginal changes of effort from the implemented effort level (i.e., if $d\phi^s_{\widehat{e}}/de = 0$), and if $\frac{\int_{z}^{\overline{q}} f'_e(q|\widehat{e},s)dq}{\int_{z}^{\overline{q}} f(q|\widehat{e},s)dq}$ is not a function of the signal s at the threshold output z^{**} given the equilibrium effort $\widehat{e}$, then the wage does not depend on the realization of the signal s.