Political Economy in a Changing World[*]

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June 2013

Abstract

We provide a general framework for the analysis of the dynamics of institutional change (e.g., democratization, extension of political rights or repression of different groups), and how these dynamics interact with (anticipated and unanticipated) changes in the distribution of political power and in economic structure. We focus on the Markov Voting Equilibria, which require that economic and political changes should take place if there exists a subset of players with the power to implement such changes and who will obtain higher expected discounted utility by doing so. Assuming that economic and political institutions as well as individual types can be ordered, and preferences and the distribution of political power satisfy natural "single crossing" (increasing differences) conditions, we prove the existence of a pure-strategy equilibrium, provide conditions for its uniqueness, and present a number of comparative static results that apply at this level of generality. We then use this framework to study the dynamics of political rights and repression in the presence of radical groups that can stochastically grab power. We characterize the conditions under which the presence of radicals leads to repression (of less radical groups), show a type of path dependence in politics resulting from radicals coming to power, and identify a novel strategic complementarity in repression.

Keywords: Markov Voting Equilibrium, dynamics, median voter, stochastic shocks, extension of franchise, repression.

JEL Classification: D71, D74, C71.

[*]An earlier draft was circulated under the title "Markov Voting Equilibria: Theory and Applications". We thank participants of Wallis Institute Annual Conference, CIFAR meeting in Toronto, and of seminars at Georgetown, ITAM, Northwestern, London School of Economics, Stanford, UPenn, Warwick and Zurich for helpful comments.

1 Introduction

Political change often takes place in the midst of uncertainty and turmoil, which sometimes brings to power — or paves the road for the rise of — the most radical factions, such as the militant Jacobins during the Reign of Terror in the French Revolution or the Nazis during the crisis of the Weimar Republic. The possibility of "extreme" outcomes is of interest not only because the resulting regimes have caused much human suffering and powerfully shaped the course of history, but also because, in many episodes, the fear of such radical extremist regimes has been one of the drivers of repression against a whole gamut of opposition groups. The events leading up to the October Revolution of 1917 in Russia illustrate both how an extremist fringe group can ascend to power, and the dynamics of repression partly motivated by the desire of ruling elites to prevent the empowerment of extremist — and sometimes also of more moderate — elements.

Russia entered the 20$^{\text{th}}$ century as an absolute monarchy, but started a process of limited political reforms in response to labor strikes and civilian unrest in the aftermath of its defeat in the Russo-Japanese war of 1904-1905. Despite the formation of political parties (for the first time in Russian history) and an election with a wide franchise, the repression against the regime's opponents continued, and the parliament, the Duma, had limited powers and was considered by the tsar as an advisory rather than legislative body (Pipes, 1995). The tsar still retained control, in part relying on repression against the leftist groups, his veto power, the right to dissolve the Duma, full control of the military and cabinet appointments, and his ability to rule by decree when the Duma was not in session. This may have been partly motivated by the fear of further strengthening the two major leftist parties, Social Revolutionaries and Social Democrats (corresponding to communists, consisting of the Bolsheviks and the Mensheviks), which together controlled about 2/5 of the 1906 Duma and explicitly targeted a revolution.[1]

World War I, which became very unpopular following large casualties and territorial losses, created the opening for the Bolsheviks, bringing to power the Provisional Government in the February Revolution of 1917, and then later, the moderate Social Revolutionary Alexander

[1] Lenin, the leader of the Bolshevik wing of the Social Democrats, recognized that a revolution was possible only by exploiting turmoil. In the context of the 1906 Duma, he stated: "Our task is [. . .] to use the conflicts within this Duma, or connected with it, for choosing the right moment to attack the enemy, the right moment for an insurrection against the autocracy." Later, he argued: "[. . .] the Duma should be used for the purposes of the revolution, should be used mainly for promulgating the Party's political and socialist views and not for legislative 'reforms,' which, in any case, would mean supporting the counter-revolution and curtailing democracy in every way."

Kerensky. Additional military defeats of the Russian army in the summer of 1917, the destruction of the military chain of command by Bolshevik-led soldier committees, and Kerensky's willingness to enter into an alliance with Social Democrats to defeat the attempted coup by the army during the Kornilov affair strengthened the Bolsheviks further. Though in the elections to the Constituent Assembly in November 1917, they had only a small fraction of the vote, the Bolsheviks successfully exploited their control of Petrograd Soviets to outmaneuver the more popular Social Revolutionaries, first entering into an alliance with so-called Left Social Revolutionaries, and then coercing them to leave the government so as to form their own one-party dictatorship.

This episode thus illustrates both the possibility of a series of transitions bringing to power some of the most radical groups and the potential implications of the concerns of moderate political transitions further empowering radical groups. Despite a growing literature on political transitions, the issues we have just illustrated in the context of the Bolshevik Revolution cannot be studied with existing models,[2] because they necessitate a dynamic model where several groups can form temporary coalitions and a rich set of stochastic shocks creates a changing environment, potentially leading to a sequence of political transitions away from current powerholders. Such a model could also shed further light on key questions in the literature on regime transitions, including those concerning political transitions with several heterogeneous groups, gradual enfranchisement, and the interactions between regime dynamics and coalition formation. In this paper, we develop a framework for the study of dynamic political economy in the presence of stochastic shocks and changing environments, which we then apply to an analysis of the implications of potential shifts of power to radical groups during tumultuous times. The next example provides a first glimpse of the type of abstraction we will use.

Example 1 Consider a society consisting of n groups, spanning from $-l < 0$ (left-wing) to $r > 0$ (right-wing), with group 0 normalized to contain the median voter. For concreteness, suppose that $n = 3$, and that the rightmost player corresponds to the Russian tsar, the middle player to moderate groups, and the leftmost group to Bolsheviks. The stage payoff of each group depends on current policies, which are determined by the politically powerful coalition in the current "political state". Suppose that there are $2n - 1$ political states, each state specifying

[2] These types of political dynamics are not confined to episodes in which extreme left groups might come to power. The power struggles between secularists and religious groups in Turkey and more recently in the Middle East and North Africa are also partly motivated by concerns on both sides that political power will irrevocably — or at least persistently — shift to the other side.

which of the "extreme" players are repressed and excluded from political decision-making. With $n = 3$, the five states are $s = 2$ (both moderates and Bolsheviks are repressed and the tsar is the dictator), 1 (Bolsheviks are repressed), 0 (nobody is repressed and power lies with moderates), -1 (the tsar is repressed or eliminated), and finally -2 (the tsar and moderates are repressed, i.e. a Bolshevik dictatorship). Since current policies depend on the political state, we can directly define stage payoffs as a function of the current state for each player, $u_i(s)$ (which is inclusive of repression costs, if any). Suppose that starting in any state $s \neq -2$, a stochastic shock can bring the Bolsheviks to power and this shock is more likely when s is lower.

In addition to proving the existence and characterizing the structure of pure-strategy equilibria, our framework enables us to establish the following types of results. First, in the absence of stochastic shocks bringing Bolsheviks to power, $s = 0$ (no repression or democracy) is stable in the sense that moderates would not like to initiate repression, but $s > 0$ may also be stable, because the tsar may prefer to incur the costs of repression to implement policies more in line with his preferences. Second, and more interestingly, moderates may also initiate repression starting with $s = 0$ if there is the possibility of a switch of power to Bolsheviks. Third, and paradoxically, the tsar may be more willing to grant political rights to moderates when Bolsheviks are stronger, because this might make a coalition between the latter two groups less likely (this is an illustration of what we refer to as "slippery slope" considerations and shows the general non-monotonicities in our model: when Bolsheviks are stronger, the tsar has less to fear from the slippery slope). Fourth, there is history dependence in the sense that once Bolsheviks come to power and leave power, a new (different) stable state may emerge. Finally, there is strategic complementarity in repression: the anticipation of repression by Bolsheviks encourages repression by moderates and the tsar.[3]

Though stylized, this example communicates the complex strategic interactions involved in dynamic political transitions in the presence of stochastic shocks and changing environments. Against this background, the framework we develop will show that, under natural assumptions, we can characterize the equilibria of this class of environments fairly tightly and perform comparative statics, shedding light on these and a variety of other dynamic strategic interactions.

[3] This result is also interesting as it provides a new perspective on why repression may differ markedly across societies. For example, Russia before the Bolshevik Revolution repressed the leftists, and after the Bolshevik Revolution systematically repressed the rightists and centrists, while the extent of repression of either extreme has been more limited in the United Kingdom. Such differences are often ascribed to differences in "political culture". Our result instead suggests that (small) differences in economic interests or political costs of repression can lead to significantly different repression outcomes.

Formally, we consider a generalization of the environment discussed in the example. Society consists of $i = 1, 2, ..., n$ players (groups or individuals) and $s = 1, 2, ..., m$ *states*, which represent both different economic arrangements with varying payoffs for different players, and different political arrangements and institutional choices. Stochastic shocks are modeled as stochastic changes in *environments*, which contain information on preferences of all players over states and the distribution of political power within states. This approach is general enough to capture a rich set of permanent and transitory (as well as both anticipated or unanticipated) stochastic shocks depending on the current state and environment. Players care about the expected discounted sum of their utility, and they make joint choices among feasible political transitions, based on their political power. Our key assumption is that both preferences and the distribution of political power satisfy a natural *single-crossing* (increasing differences) property: we assume that players and states are "ordered," and higher-indexed players relatively prefer higher-indexed states and also tend to have greater political power in such states. (Changes in environments shift these preferences and distribution of political power, but maintain increasing differences).

Our notion of equilibrium is *Markov Voting Equilibrium (MVE)*, which comprises two natural requirements: (1) that changes in states should take place if there exists a subset of players with the power to implement them and who will obtain higher continuation utility (along the equilibrium path) by doing so; (2) that strategies and continuation utilities should only depend on payoff-relevant variables and states. Under these assumptions, we establish the existence of pure-strategy equilibria. Furthermore, we show that the stochastic path of states in any MVE ultimately converges to a limit state — i.e., to a state that does not induce further changes once reached, though this limit state may depend on the exact timing and sequence of shocks (Theorems 1 and 3).[4] Although MVE are not always unique, we also provide sufficient conditions that ensure uniqueness (Theorems 2 and 4). We further demonstrate a close correspondence between these MVE and the pure-strategy Markov perfect equilibria of our environment (Theorem 5).

Despite the generality of the framework described here and the potential countervailing forces highlighted by our example above, we also establish a number of comparative static results. Here we only mention one of them. Consider a change in environment which leaves preferences or the allocation of political power in any of the states $s = 1, ...s'$ unchanged, but potentially changes them in states $s = s' + 1, ..., m$. The result is that if the steady state of equilibrium

[4] This last result also implies that, in contrast to many other models of institutional persistence, ours features "true path dependence" as defined, for example, by Page (2006), who criticizes many existing models of "path dependence" for being invariant to the sequencing of shocks.

dynamics described above, x, did not experience change (i.e., $x \leq s'$), then the new steady state emerging after the change in environment can be no smaller than this steady state (Theorem 6). Intuitively, before the change, a transition to any of the smaller states $s \leq x$ could have been chosen, but was not. Now, given that preferences and political power did not change for these states, they have not become more attractive.[5] An interesting and novel implication of this result is that in some environments, there may exist *critical states*, such as a "sufficiently democratic constitution," and if these critical states are reached before the arrival of certain major shocks or changes (which might have otherwise led to their collapse), there will be no turning back (see Corollary 1). This result provides a different interpretation of the durability of certain democratics regimes than the approaches based on "democratic capital" (e.g., Persson and Tabellini, 2009): a democracy will survive forever if it is not shocked or challenged severely while still progressing towards the "sufficiently democratic constitution/state", but will fall if there is a shock before this state is reached.

The second part of the paper applies our framework to the emergence and implications of radical politics. After establishing that our framework and comparative statics can be directly applied to the class of problems described in Example 1, we derive a number of additional results for this application, some of which were outlined above.

Our paper is related to a large political economy literature. First, our previous work, in particular Acemoglu, Egorov, and Sonin (2012), takes one step in this direction by introducing a model for the analysis of the dynamics and stability of different political rules and constitutions. However, that approach not only heavily relies on deterministic and stationary environments (thus ruling out changes in political power or preferences) but also focuses on environments in which the discount factor is sufficiently close to 1 so that all agents just care about the payoff from a stable state (that will emerge and persists) if such a state exists. Here, in contrast, it is crucial that political change and choices are motivated by the entire path of payoffs.[6]

Second, several papers on dynamic political economy and on dynamics of clubs emerge as

[5] In contrast, some of the higher-ranked states may have become more attractive, which may induce a transition to a higher state. In fact, perhaps somewhat surprisingly, transition to a state $s \geq s' + 1$ can take place even if all states $s = s' + 1, ..., m$ become less attractive for all agents in society.

[6] In Acemoglu, Egorov and Sonin (2010), we studied political selection and government formation in a population with heterogeneous abilities and allowed stochastic changes in the competencies of politicians. Nevertheless, this was done under two assumptions, which significantly simplified the analysis and made it much less applicable. In particular, stochastic shocks were assumed to be very infrequent and the discount factor was taken to be close to 1. Acemoglu, Egorov and Sonin (2011) took a first step towards introducing stochastic shocks, but only confined to the exogenous emergence of new extreme states (and without any of the general characterization or comparative static results presented here).

special cases of our paper. Among these, Roberts (1999) deserves special mention as an important precursor of our analysis. Roberts studies a dynamic model of club formation in which current members of the club vote about whether to admit new members or exclude some of the existing ones. Roberts focuses on a limited set of transitions, also makes single-crossing type assumptions and only considers non-stochastic environments and majoritarian voting (see also Barberà, Maschler, and Shalev, 2001, for a related setup). Both our framework and characterization results are more general, not only because they incorporate stochastic elements and more general distributions of political power, but also because we provide conditions for uniqueness, convergence to steady states, and general comparative static results. In addition, Gomes and Jehiel's (2005) paper, which studies dynamics in a related environment with side transfers, is also noteworthy. This paper, however, does not include stochastic elements or similar general characterization results either. Strulovici (2010), who studies a voting model with stochastic arrival of new information, is also related, but his focus is on information leading to inefficient dynamics, while changes in political institutions or voting rules are not part of the model.

Third, our motivation is also related to the literature on political transitions. Acemoglu and Robinson (2000a, 2001) consider environments in which institutional change is partly motivated by a desire to reallocate political power in the future to match the current distribution of power.[7] Acemoglu and Robinson's analysis is simplified by focusing on a society consisting of two social groups (and in Acemoglu and Robinson, 2006, with three social groups). In Acemoglu and Robinson (2001), Fearon (2004), Powell (2006), Hirshleifer, Boldrin and Levine (2009), and Acemoglu, Ticchi, and Vindigni (2010), anticipation of future changes in political power leads to inefficient policies, civil war, or collapse of democracy. There is a growing literature that focuses on situations where decisions of the current policy makers affect the future allocation of political power (see also, Besley and Coate, 1998).

Fourth, there is a small literature on strategic use of repression, which includes Acemoglu and Robinson (2000b), Gregory, Schroeder, and Sonin (2011) and Wolitzky (2011). In Wolitzky (2011), different political positions (rather than different types of players) are repressed in order to shift the political equilibrium in the context of a two-period model of political economy. In Acemoglu and Robinson (2000b), repression arises because political concessions can be interpreted as a sign of weakness. None of the papers discussed in the previous three paragraphs

[7] Other related contributions here include Alesina, Angeloni, and Etro (2005), Barberà and Jackson (2004), Messner and Polborn (2004), Bourguignon and Verdier (2000), Burkart and Wallner (2000), Jack and Lagunoff (2008), Lagunoff (2006), and Lizzeri and Persico (2004).

study the issues we focus on or make progress towards a general framework of the sort presented here.

The rest of the paper is organized as follows. In Section 2, we present our general framework and introduce the concept of MVE. Section 3 contains the analysis of MVE. We start with the stationary case (without shocks), then extend the analysis to the general case where shocks are possible, and then compare the concepts of MVE to Markov Perfect Equilibrium in a properly defined dynamic game. We also establish several comparative static results that hold even at this level of generality; this allows us to study the society's reactions to shocks in applied models. Section 4 applies our framework to the study of radical politics. Section 5 discusses a number of extensions. Section 6 concludes.

2 General Framework

Time is discrete and infinite, indexed by $t \geq 1$. The society consists of n *players* (representing individuals or groups), $N = \{1, \ldots, n\}$. The set of players is ordered, and the order reflects the initial distribution of some variable of interest. For example, higher-indexed players may be richer, or more pro-authoritarian, or more right-wing on social issues. In each period, the society is in one of the h *environments* $\mathcal{E} = \{E_1, \ldots, E_h\}$, which determine preferences and the distribution of political power in society (as described below). We model stochastic elements by assuming that, at each date, the society transitions from environment E to environment E' with probability $\pi(E, E')$. Naturally, $\sum_{E' \in \mathcal{E}} \pi(E, E') = 1$. We assume:

Assumption 1 ***(Ordered Transitions)*** *If* $1 \leq x < y \leq h$*, then*

$$\pi(E_y, E_x) = 0.$$

Assumption 1 implies that there can only be at most a finite number of shocks. It also stipulates that environments are numbered so that only transitions to higher-numbered environments are possible.[8] Though this is without loss of generality, it enables us to use the convention that once the last environment, E_h, has been reached, there will be no further stochastic shocks.[9]

We model preferences and the distribution of political power by means of *states*, belonging to a finite set $S = \{1, \ldots, m\}$.[10] The set of states is *ordered*: loosely speaking, this will generally

[8] Assumption 1 does not preclude the possibility that the same environment will recur several times. For example, the possibility of q transitions between E_1 and E_2 can be modeled by setting $E_3 = E_1$, $E_4 = E_2$, etc.

[9] This does not mean that the society must reach E_h on every path: for example, it is permissible to have three environments with $\pi(E_1, E_2) = \pi(E_1, E_3) > 0$, and all other transition probabilities equal to zero.

[10] The implicit assumption that the set of states is the same for all environments is without any loss of generality.

imply that higher-indexed states provide both greater economic payoffs and more political power to higher-indexed players. An example would be a situation in which higher-indexed states correspond to more non-democratic arrangements, which are both economically and politically better for richer, more elite groups. The payoff of player $i \in N$ in state $s \in S$ and environment $E \in \mathcal{E}$ is $u_{E,i}(s)$.

To capture relative preferences and power of players in different states, we will frequently make use of the following definition:

Definition 1 ***(Increasing Differences)*** *Vector $\{w_i(s)\}_{i \in A}^{s \in B}$, where $A, B \subset \mathbb{R}$, satisfies the* increasing differences *condition if for any agents $i, j \in A$ such that $i > j$ and any states $x, y \in B$ such that $x > y$,*

$$w_i(x) - w_i(y) \geq w_j(x) - w_j(y).$$

The following is one of our key assumptions:

Assumption 2 ***(Increasing Differences in Payoffs)*** *In every environment $E \in \mathcal{E}$, the vector of utility functions, $\{u_{E,i}(s)\}_{i \in N}^{s \in S}$, satisfies the increasing differences condition.*

Note that payoffs $\{u_{E,i}(s)\}$ are directly assigned to combinations of states and environments. An alternative would be to assign payoffs to some other actions, e.g., "policies", which are then determined endogenously by the same political process that determines transitions between states. This is what we do in Section 4, and as our analysis there shows, under fairly weak conditions, the current state will determine the choice of action (policy), so payoffs will then be indirectly defined over states and environments. Here we are thus reducing notation by directly writing them as $\{u_{E,i}(s)\}$.

We model the distribution of political power in a state using the notion of *winning coalitions*. This captures information on which subsets of agents have the (political) power to implement economic or political change, here corresponding to a transition from one state to another. We denote the set of winning coalitions in state s and environment E by $W_{E,s}$, and impose the following standard assumption:

Assumption 3 ***(Winning Coalitions)*** *For environment $E \in \mathcal{E}$ and state $s \in S$, the set of winning coalitions $W_{E,s}$ satisfies:*

1. *(monotonicity) if $X \subset Y \subset N$ and $X \in W_{E,s}$, then $Y \in W_{E,s}$;*

2. *(properness) if* $X \in W_{E,s}$*, then* $N \setminus X \notin W_{E,s}$*;*

3. *(decisiveness)* $W_{E,s} \neq \varnothing$.

The first part of Assumption 3 states that if some coalition has the capacity to implement change, then a larger coalition also does. The second part ensures that if some coalition has the capacity to implement change, then the coalition of the remaining players (its complement) does not (effectively ruling out "submajority rule"). Finally, the third part, in the light of monotonicity propery, is equivalent to $N \in W_{E,s}$, and thus states that if all players want to implement a change, they can do so. Several common models of political power are special cases. For example, if a player is a dictator in some state, then the winning coalitions in that state are all those that include him; if we need unanimity for transitions, then the only winning coalition is N; if there is majoritarian voting in some state, then the set of winning coalitions consists of all coalitions with an absolute majority of the players.

Assumption 3 puts minimal and natural restrictions on the set of winning coalitions $W_{E,s}$ in each given state $s \in S$. Our main restriction on the distribution of political power will be, as discussed in the Introduction, the requirement of some "monotonicity" of political power — that higher-indexed players have no less political power in higher-indexed states. To formally formulate this restriction, we need the notion of a quasi-median voter (see Acemoglu, Egorov, and Sonin, 2012).

Definition 2 ***(Quasi-Median Voter)*** *Player ranked* i *is a* quasi-median voter *(QMV) in state* s *(in environment* E*) if for any winning coalition* $X \in W_{E,s}$*,* $\min X \leq i \leq \max X$*.*

Let $M_{E,s}$ denote the set of QMVs in state s in environment E. Then by Assumption 3, $M_{E,s} \neq \varnothing$ for any $s \in S$ and $E \in \mathcal{E}$; moreover, the set $M_{E,s}$ is connected: whenever $i < j < k$ and $i, k \in M_{E,s}$, $j \in M_{E,s}$. In many cases, the set of quasi-median voters is a singleton, $|M_{E,s}| = 1$. Examples include: one player is the dictator, i.e., $X \in W_{E,s}$ if and only if $i \in X$ (and then $M_{E,s} = \{i\}$), or majoritarian voting among sets containing odd numbers of players, or there is a weighted majority in voting with "generic weights" (see Section 4). An example where $M_{E,s}$ is not a singleton is the unanimity rule.

The following assumption ensures that the distribution of political power is "monotone" over states.

Assumption 4 ***(Monotone Quasi-Median Voter Property, MQMV)*** *In any environment $E \in \mathcal{E}$, the sequences $\{\min M_{E,s}\}_{s \in S}$ and $\{\max M_{E,s}\}_{s \in S}$ are nondecreasing in s.*

The essence of Assumption 4 is that political power (weakly) shifts towards higher-indexed players in higher-indexed states. For example, if a certain number of higher-indexed players are powerful enough to implement a transition in some state, then they are also sufficiently powerful to do so in a higher-indexed state. This would hold in a variety of applications, including the one we present in Section 4 and Roberts's (1999) model. Trivially, if $M_{E,s}$ is a singleton in every state, it is equivalent to $M_{E,s}$ being nondecreasing (where $M_{E,s}$ is treated as the single element).

For some applications, one might want to restrict feasible transitions between states that the society may implement; for example, it might be realistic to assume that only transitions to adjacent states are possible. To incorporate such possibilities, we introduce the mapping $F = F_E : S \to 2^S$, which maps every $x \in S$ into the set of states to which society may transition. In other words, $y \in F_E(x)$ means that the society may transition from x to y in environment E. We do not assume that $y \in F_E(x)$ implies $x \in F_E(y)$, so certain transitions may be irreversible. We impose:

Assumption 5 ***(Feasible Transitions)*** *For each environment $E \in \mathcal{E}$, F_E satisfies:*

1. *For any $x \in S$, $x \in F_E(x)$;*

2. *For any states $x, y, z \in S$ such that $x < y < z$ or $x > y > z$: If $z \in F_E(x)$, then $y \in F_E(x)$ and $z \in F_E(y)$.*

The key requirement, encapsulated in the second part, is that if a transition between two states is feasible, then any transitions (in the same direction) between intermediate states are also feasible. Special cases of this assumption include: (a) any transition is possible: $F_E(x) = S$ for any x and E; (b) one-step transitions: $y \in F_E(x)$ if and only if $|x - y| \leq 1$; (c) one directional transitions: $y \in F_E(x)$ if and only if $x \leq y$.[11]

Finally, we assume that the discount factor, $\beta \in [0, 1)$, is the same for all players and across all environments. To recap, the full description of each environment $E \in \mathcal{E}$ is given by a tuple $\left(N, S, \beta, \{u_{E,i}(s)\}_{i \in N}^{s \in S}, \{W_{E,s}\}_{s \in S}, \{F_E(s)\}_{s \in S}\right)$.

[11] In an earlier version, we also allowed for costs of transitions between states, which we now omit to simplify the exposition.

Each period t starts with environment $E_{t-1} \in \mathcal{E}$ and with state s_{t-1} inherited from the previous period; Nature determines E_t with probability distribution $\pi(E_{t-1}, E_t)$, and then the players decide on the transition to any feasible s_t as we describe next. We take $E_0 \in \mathcal{E}$ and $s_0 \in S$ as given. At the end of period t, each player receives the stage payoff

$$v_i^t = u_{E_t,i}(s_t). \tag{1}$$

Denoting the expectation at time t by $\mathbb{E}_t$, the expected discounted payoff of player i by the end of period t can be written as

$$V_i^t = \mathbb{E}_t \sum\nolimits_{k=0}^{\infty} \beta^k u_{E_{t+k},i}(s_{t+k}).$$

The timing of events within each period is:

1. The environment E_{t-1} and state s_{t-1} are inherited from period $t-1$.
2. There is a change in environment from E_{t-1} to $E_t \in \mathcal{E}$ with probability $\pi(E_{t-1}, E_t)$.
3. Society (collectively) decides on state s_t, subject to $s_t \in F_E(s_{t-1})$.
4. Each player gets stage payoff given by (1).

We omit the exact sequence of moves determining transitions across states (in step 3) as this is not required for the Markov Voting Equilibrium (MVE) concept. The exact game form is introduced when we study the noncooperative foundations of MVE.[12]

MVE will be characterized by a collection of transition mappings $\phi = \{\phi_E : S \to S\}_{E \in \mathcal{E}}$. We let ϕ_E^k be the k^{th} iteration of ϕ_E (with $\phi_E^0(s) = s$). With ϕ, we can associate continuation payoffs $V_{E,i}^{\phi}(s)$ for player i in state s and environment E, which are recursively given by

$$V_{E,i}^{\phi}(s) = u_{E,i}(s) + \beta \sum_{E' \in \mathcal{E}} \pi(E, E') V_{E',i}^{\phi}(\phi_{E'}(s)). \tag{2}$$

As $0 \le \beta < 1$, the values $V_{E,i}^{\phi}(s)$ are uniquely defined by (2).

Definition 3 ***(Markov Voting Equilibrium, MVE)*** *A collection of transition mappings* $\phi = \{\phi_E : S \to S\}_{E \in \mathcal{E}}$ *is a* Markov Voting Equilibrium *if the following three properties hold:*

1. *(feasibility) for any environment $E \in \mathcal{E}$ and for any state $x \in S$, $\phi_E(x) \in F_E(x)$;*

[12] In what follows, we use MVE both for the singular (Markov Voting Equilibrium) and plural (Markov Voting Equilibria).

2. *(core) for any environment $E \in \mathcal{E}$ and for any states $x, y \in S$ such that $y \in F_E(x)$,*

$$\left\{ i \in N : V_{E,i}^{\phi}(y) > V_{E,i}^{\phi}(\phi_E(x)) \right\} \notin W_{E,x}; \tag{3}$$

3. *(status quo persistence) for any environment $E \in \mathcal{E}$ and for any state $x \in S$,*

$$\left\{ i \in N : V_{E,i}^{\phi}(\phi_E(x)) \geq V_{E,i}^{\phi}(x) \right\} \in W_{E,x}.$$

Property 1 requires that MVE involves only feasible transitions (in the current environment). Property 2 is satisfied if no (feasible) alternative $y \neq \phi(x)$ is supported by a winning coalition in x over $\phi_E(x)$ prescribed by the transition mapping ϕ_E. This is analogous to a "core" property: no alternative should be preferred to the proposed transition by some "sufficiently powerful" coalition of players; otherwise, the proposed transition would be blocked. Of course, in this comparison, players should focus on continuation utilities, which is what (3) imposes. Property 3 requires that it takes a winning coalition to move from any state to some alternative —i.e., to move away from the status quo. This requirement singles out the status quo if there is no alternative strictly preferred by some winning coalition.

In addition, we say ϕ_E is *monotone* if for all $x, y \in S$ such that $x \geq y$, we have $\phi_E(x) \geq \phi_E(y)$ (ϕ is monotone if each of the ϕ_E's is monotone). For now, we focus on monotone MVE, i.e., MVE with monotone transition mappings for each $E \in \mathcal{E}$. In many cases this is without loss of generality, and Theorem 9 states mild sufficient conditions for when all MVE are (generically) monotone. We also refer to any state x such that $\phi_E(x) = x$ as a *steady state* or *stable* in E.

In what follows, with some abuse of notation, we will often suppress the reference to the environment and use, e.g., $u_i(s)$ instead of $u_{E,i}(s)$ or ϕ instead of ϕ_E, when this causes no confusion.

3 Analysis

In this section, we analyze the structure of MVE. We first prove existence of monotone MVE in a stationary (deterministic) environment. We then extend these results to situations in which there are stochastic shocks and nonstationary elements. After establishing the relationship between MVE and Markov Perfect Equilibria (MPE) of a dynamic game representing the framework of Section 2, we present a number of comparative static results for our general model.

3.1 Nonstochastic environment

We first study the case without any stochastic shocks, or equivalently the case of only one environment ($|\mathcal{E}| = 1$) and thus suppress the subscript E.

For any mapping $\phi : S \to S$, the continuation utility of player i after a transition to s has taken place is given by

$$V_i^{\phi}(s) = u_i(s) + \sum\nolimits_{k=1}^{\infty} \beta^k u_i\left(\phi^k(s)\right). \tag{4}$$

We start our analysis with several lemmas which will form the basis of our main results. The next one emphasizes the role that the notion of quasi-median voters (QMV) plays in our theory.

Lemma 1 *Suppose that vector $\{w_i(s)\}$ satisfies increasing differences for $S' \subset S$. Take $x, y \in S'$, $s \in S$ and $i \in N$ and let*

$$P = \{i \in N : w_i(y) > w_i(x)\}.$$

Then $P \in W_s$ if and only if $M_s \subset P$. A similar statement is true for relations $\geq$, $<$, $\leq$.

Lemma 1 is a consequence of the following reasoning: From increasing differences in payoffs, if $w_i(y) > w_i(x)$ for members of W_s, then this holds for all $i \leq \max M_s$ if $y < x$ and for all $i \geq \min M_s$ if $y > x$. In either case, this establishes the "if" part of the lemma. The "only if" part also follows from increasing differences: $w_i(y) > w_i(x)$ must hold for a connected coalition, and therefore it holds for all members of M_s (from Definition 2).

For each $s \in S$, let us introduce the binary relation $>_s$ on the set of n-dimensional vectors to designate that there exists a winning coalition in s strictly preferring one payoff vector to another. Formally:

$$w^1 >_s w^2 \Leftrightarrow \left\{i \in N : w_i^1 > w_i^2\right\} \in W_s.$$

The relation $\geq_s$ is defined similarly. Lemma 1 now implies that if a vector $\{w_i(x)\}$ satisfies increasing differences, then for any $s \in S$, the relations $>_s$ and $\geq_s$ are transitive on $\{w_\cdot(x)\}_{x \in S}$.

Our next result is critical for the rest of our analysis, establishing that, under Assumption 2 and 5, when ϕ is monotone, then continuation utilities $\left\{V_i^{\phi}(s)\right\}_{i \in N}^{s \in S}$ satisfy increasing differences.

Lemma 2 *For a mapping $\phi : S \to S$, the vector $\left\{V_i^{\phi}(s)\right\}_{i \in N}^{s \in S}$, given by (4), satisfies increasing differences if*

1. *ϕ is monotone; or*

2. for all $x \in S$, $|\phi(x) - x| \leq 1$.

This result is at the root of the central role of QMVs in our model. As is well known, median voter type results do not generally apply with multidimensional policy choices. Since our players are effectively voting over infinite dimensional choices (a sequence of policies), a natural conjecture would have been that such results would not apply in our setting either. The reason they do has a similar intuition to why voting sequentially over two dimensions of policy, over each of which preferences satisfy single crossing (increasing differences) or single peakedness, does lead to the median voter type outcomes. By backward induction, the second vote has a well-defined median voter, and then given this choice, the median voter over the first one can be determined. Loosely speaking, our recursive formulation of today's value enables us to apply this reasoning between the vote today and the vote tomorrow, and the fact that continuation utilities satisfy increasing differences is the critical step in this argument.

For mapping ϕ to constitute a MVE, it must satisfy the three properties of Definition 3. Of these, the "core" property is the most substantive one. The next lemma simplifies the analysis considerably by proving that if for a monotone mapping ϕ the core property is violated (i.e., there is a deviation that makes all members of some winning coalition in the current state better off), then one can find a *monotone deviation*—i.e., a valid deviation such that the resulting mapping after the deviation is also monotone. We call this result the Monotone Deviation Principle with analogy to the One-Stage Deviation Principle in extensive form games, which also simplifies the set of deviations one has to consider (because if some deviation makes a player better off, then there is a one-stage deviation which also does so).

Lemma 3 ***(Monotone Deviation Principle)*** *Suppose that $\phi : S \to S$ is feasible (part 1 of Definition 3) and monotone but the core property is violated in the sense that for some $x, y \in S$ (such that $y \in F(x)$),*

$$V^{\phi}(y) >_x V^{\phi}(\phi(x)) . \tag{5}$$

Then there exist $x, y \in S$ such that $y \in F(x)$, (5) *still holds, and the mapping $\phi' : S \to S$ given by*

$$\phi'(s) = \begin{cases} \phi(s) & \text{if } s \neq x \\ y & \text{if } s = x \end{cases} \tag{6}$$

is monotone.

With the help of the Monotone Deviation Principle, we can prove the following result, which will be used to establish the existence of MVE.

Lemma 4 ***(No Double Deviation)*** *Let $a \in [1, m-1]$, and let $\phi_1 : [1,a] \to [1,a]$ and $\phi_2 : [a+1, m] \to [a+1, m]$ be two monotone mappings which are MVE on their respective domains. Let $\phi : S \to S$ be defined by*

$$\phi(s) = \begin{cases} \phi_1(s) & \text{if } s \leq a \\ \phi_2(s) & \text{if } s > a \end{cases} \tag{7}$$

Then exactly one of the following is true:

1. *ϕ is a MVE on S;*

2. *there is $z \in [a+1, \phi(a+1)]$ such that $z \in F(a)$ and $V^{\phi}(z) >_a V^{\phi}(\phi(a))$;*

3. *there is $z \in [\phi(a), a]$ such that $z \in F(a+1)$ and $V^{\phi}(z) >_{a+1} V^{\phi}(\phi(a+1))$.*

Intuitively, this lemma states that if we split the set of states into two subsets, $[1, a]$ and $[a+1, m]$, and find (by induction) the MVE on these respective domains, then the combined mapping may fail to be an MVE only if either a winning coalition in a prefers to move to some (feasible) state in $[a+1, m]$, or a winning coalition in $a+1$ prefers to move to some state in $[1, a]$. But crucially, these two possibilities are mutually exclusive — a result which we use to prove our next theorem, establishing the existence of MVE.

Theorem 1 ***(Existence)*** *There exists a monotone MVE. Moreover, if ϕ is a monotone MVE, then the equilibrium path $s_0, s_1 = \phi(s_1), s_2 = \phi(s_2), \ldots$ is monotone, and there exists a limit state $s_\tau = s_{\tau+1} = \ldots = s_\infty$.*

We now provide a brief sketch of the proof of this theorem which is by induction on the number of states (here we assume for simplicity that all transitions are feasible). If $m = 1$, then $\phi : S \to S$ given by $\phi(1) = 1$ is an MVE for trivial reasons. For $m > 1$, we assume, to obtain a contradiction, that there is no MVE. Take any of $m-1$ possible splits of S into nonempty $C_a = \{1, \ldots, a\}$ and $D_a = \{a+1, \ldots, m\}$, where $a \in \{1, \ldots, m-1\}$, and then take MVE ϕ_1^a on C_a and MVE ϕ_2^a on D_a (assume for simplicity that they are unique; the Appendix describes the way we select ϕ_1^a and ϕ_2^a in the general case). Lemma 4 implies that either there is a deviation from a to $[a+1, \phi_2^a(a+1)]$ or a deviation from $a+1$ to $[\phi_1^a(a), a]$, but not both. Denote $g(a) = r$ (for "right") in the former case, and $g(a) = l$ in the latter. Then g is a well-defined single-valued function. We then have the following possibilities.

If $g(1) = r$, we can "extend" the MVE ϕ_2^1 onto the entire domain by assigning $\phi(1) \in [2, m]$ appropriately; similarly, if $g(m-1) = l$, we can extend ϕ_1^{m-1} by choosing $\phi(m) \in [1, m-1]$

appropriately (the details are provided in the Appendix). It remains to consider the case where $g(1) = l$ and $g(m-1) = r$. Then there must exist $a \in \{2, \ldots, m-1\}$ such that $g(a-1) = l$ and $g(a) = r$. We take equilibria ϕ_1^{a-1} on $[1, a-1]$ and ϕ_2^a on $[a+1, m]$, and consider $\phi : S \to S$ given by

$$\phi(s) = \begin{cases} \phi_1^{a-1}(s) & \text{if } s < a \\ b & \text{if } s = a \\ \phi_2^a(s) & \text{if } s > a \end{cases},$$

where $b \in \left[\phi_1^{a-1}(a-1), a-1\right] \cup \left[a+1, \phi_2^a(a+1)\right]$ is picked so that $V_i^\phi(b)$ is maximized for some $i \in M_a$ (and $b \in F(a)$). Suppose, without loss of generality, that $b < a$, then $\phi|_{[1,a]}$ is a MVE on $[1, a]$. By Lemma 3, to show that the core property is satisfied, it suffices to check that there is no deviation from $a+1$ to $[b, a]$; this follows from $g(a) = r$. The other two properties, feasibility and persitence, hold by construction, and thus ϕ is MVE. The Appendix fills in the details of this argument.

We next study the uniqueness of monotone MVE. We first introduce the following definitions.

Definition 4 ***(Single-Peaked Preferences)*** *Individual preferences are* single-peaked *if for every $i \in N$ there exists $x \in S$ such that whenever, for states $y, z \in S$, $z < y \leq x$ or $z > y \geq x$, $u_i(z) < u_i(y)$.*

Definition 5 ***(One-Step Transitions)*** *We say that* only one-step transitions are possible *if for any $x, y \in S$ with $|x - y| > 1$, $y \notin F(x)$.*

The next examples shows that a monotone MVE is not always unique.

Example 2 (Example with two MVE) Suppose that there are three states A, B, C, and two players 1 and 2. The decision-making rule is unanimity in all states. Payoffs are given by

id	*A*	*B*	*C*
1	20	5	10
2	10	5	20

Then, with β sufficiently close to 1 (e.g., $\beta = 0.9$), there are two MVE. In one, $\phi_1(A) = \phi_2(B) = A$ and $\phi_1(C) = C$. In another, $\phi_2(A) = A$, $\phi_2(B) = \phi_2(C) = C$. This is possible because preferences are not single-peaked, and there is more than one QMV in all states. Example 6 in the Appendix shows that making preferences single peaked is by itself insufficient to restore uniqueness.

The next theorem provides sufficient conditions for generic uniqueness of monotone MVE.

Theorem 2 ***(Uniqueness)*** *The monotone MVE is (generically) unique if*

1. *for every $s \in S$, M_s is a singleton; and/or*

2. *only one-step transitions are possible and preferences are single-peaked.*

Though somewhat restrictive, several interesting applied problems satisfy one or the other parts of the conditions of this theorem. In addition, Theorem 9 below shows that under essentially the same assumptions any MVE is monotone.

3.2 Stochastic environments

We now extend our analysis to the case in which there are stochastic shocks, which will also enable us to deal with "nonstationary" in the economic environment, for example, because the distribution of political power or economic preferences will change in a specific direction in the future. By Assumption 1, environments are ordered as $E_1, E_2, \ldots, E_h$ so that $\pi(E_x, E_y) = 0$ if $x > y$. This means that when (and if) we reach environment E_h, there will be no further shocks, and the analysis from Section 3.1 is applicable from then on. In particular, we get the same conditions for existence and uniqueness of MVE. We can now use backward induction from environment E_h to characterize equilibrium transition mappings in lower-indexed environments, essentially using Lemma 2, which established that when ϕ is monotone, continuation utilities satisfy increasing differences.

Here we outline this backward induction argument. Take an MVE ϕ_{E_h} in environment E_h (its existence is guaranteed by Theorem 1). Suppose that we have characterized an MVE $\{\phi_E\}_{E \in \{E_k, \ldots, E_h\}}$ for some $k = 1, \ldots, h-1$; let us construct ϕ_{E_k} which would make $\{\phi_E\}_{E \in \{E_k, \ldots, E_h\}}$ an MVE in $\{E_k, \ldots, E_h\}$. Continuation utilities in environment E_k are:

$$\begin{aligned} V^{\phi}_{E_k,i}(s) &= u_{E_k,i}(s) + \beta \sum_{E' \in \{E_k, \ldots, E_h\}} \pi\left(E_k, E'\right) V^{\phi}_{E',i}\left(\phi_{E'}(s)\right) \\ &= u_{E_k,i}(s) + \beta \sum_{E' \in \{E_{k+1}, \ldots E_h\}} \pi\left(E_k, E'\right) V^{\phi}_{E',i}\left(\phi_{E'}(s)\right) \\ &+ \beta \pi\left(E_k, E_k\right) V^{\phi}_{E_k,i}\left(\phi_{E_k}(s)\right). \end{aligned} \tag{8}$$

By induction, we know $\phi_{E'}$ and $V^{\phi}_{E'}\left(\phi_{E'}(s)\right)$ for $E' \in \{E_{k+1}, \ldots, E_h\}$. We next show that there exists ϕ_{E_k} that is an MVE given continuation values $\left\{V^{\phi}_{E_k,i}(s)\right\}_{s \in S}$ from (8). Denote

$$\begin{aligned} \tilde{u}_{E_k,i}(s) &= u_{E_k,i}(s) + \beta \sum_{E' \in \{E_{k+1}, \ldots, E_h\}} \pi\left(E_k, E'\right) V^{\phi}_{E',j}\left(\phi_{E'}(s)\right), \\ \tilde{\beta} &= \beta \pi\left(E_k, E_k\right) \end{aligned}$$

Then rearranging equation (8):

$$V_{E_k,i}^{\phi}(s) = \tilde{u}_{E_k,i}(s) + \tilde{\beta} V_{E_k,i}^{\phi}\left(\phi_{E_k}(s)\right).$$

Since $\{\tilde{u}_{E_k,i}(s)\}_{i \in N}^{s \in S}$ satisfy increasing differences, we can simply apply Theorem 1 to the modified environment $E = \left(N, S, \tilde{\beta}, \{\tilde{u}_{E_k,i}(s)\}_{i \in N}^{s \in S}, \{W_{E_k,s}\}_{s \in S}, \{F_{E_k}(s)\}_{s \in S}\right)$ to characterize ϕ_{E_k}. Then by definition of MVE, since $\{\phi_E\}_{E \in \{E_k,\ldots,E_h\}}$ was an MVE, we have that $\{\phi_E\}_{E \in \{E_k,\ldots E_h\}}$ is an MVE in $\{E_k, \ldots, E_h\}$, proving the desired result. Proceeding inductively we characterize an entire MVE $\phi = \{\phi_E\}_{E \in \{E_1,\ldots E_h\}}$. This argument establishes:

Theorem 3 ***(Existence)*** *There exists an MVE $\phi = \{\phi_E\}_{E \in \mathcal{E}}$. Furthermore, there exists a limit state $s_\tau = s_{\tau+1} = \ldots = s_\infty$ (with probability 1) but this limit state depends on the timing and realization of stochastic shocks and the path to a limit state need not be monotone.*

Establishing the uniqueness of MVE is more challenging because single peakedness is not necessarily inherited by continuation utilities (this is shown, for instance, by Example 7 in the Appendix). Nevertheless, the following theorem provides straightforward sufficient conditions for uniqueness.

Theorem 4 ***(Uniqueness)*** *The monotone MVE is (generically) unique if at least one of the following conditions holds:*

1. *for every environment $E \in \mathcal{E}$ and any state $s \in S$, $M_{E,s}$ is a singleton;*

2. *in each environment, only one-step transitions are possible; each player's preferences are single-peaked; and, moreover, for each state s there is a player i such that $i \in M_{E,s}$ for all $E \in \mathcal{E}$ and the peaks (for all $E \in \mathcal{E}$) of i's preferences do not lie on different sides of s.*

The first sufficient condition is the same as in Theorem 2, while the second strengthens its equivalent: it would be satisfied, for example, if players' bliss points and the distribution of political power do not change "much" as a result of shocks.

3.3 Noncooperative game

We have so far presented the concept of MVE without introducing an explicit noncooperative game. This is partly motivated by the fact that several plausible noncooperative games would underpin the notional MVE. In this section, we provide one plausible and transparent noncooperative game and formally establish the relationship between the Markov Perfect Equilibria (MPE) of this game and the set of MVE.

For each environment $E \in \mathcal{E}$ and state $s \in S$, let us introduce a *protocol* $\theta_{E,s}$, which is a finite sequence of all states in $F_s \setminus \{s\}$ capturing the order in which different transitions are considered within the period. Then the exact sequence of events in this noncooperative game is given as follows:

1. The environment E_{t-1} and state s_{t-1} are inherited from period $t-1$.

2. Environment transitions are realized: $E_t = E \in \mathcal{E}$ with probability $\pi(E_{t-1}, E)$.

3. The first alternative, $\theta_{E_t,s_{t-1}}(j)$ for $j = 1$, is voted against the status quo s. That is, all players are ordered in a sequence and must support either the "current proposal" $\theta_{E_t,s_{t-1}}(j)$ or the status quo s.[13] If the set of those who supported $\theta_{E_t,s_{t-1}}(j)$ is a winning coalition — i.e., it is in $W_{E_t,s_{t-1}}$— then $s_t = \theta_{E_t,s_{t-1}}(j)$; otherwise, this step repeats for the next j. If all alternatives have been voted and rejected for $j = 1, \ldots, |F_s| - 1$, then the new state is $s_t = s_{t-1}$.

4. Each player gets stage payoff given by (1).

We study (pure-strategy) MPE of this game. Naturally, each MPE induces an equilibrium behavior which can be represented by a set of transition mappings $\phi = \{\phi_E\}_{E \in \mathcal{E}}$. In particular, here $\phi_E(s)$ is the state to which the equilibrium play transitions starting with state s in environment E. Then we have:

Theorem 5 ***(MVE vs. MPE)***

1. *For any MVE ϕ, there exists a set of protocols $\{\theta_{E,s}\}_{E \in \mathcal{E}}^{s \in S}$ such that there exists a MPE which induces ϕ.*

2. *Conversely, if for some set of protocols $\{\theta_{E,s}\}_{E \in \mathcal{E}}^{s \in S}$ and some MPE σ, the corresponding transition mapping $\phi = \{\phi_E\}_{E \in \mathcal{E}}$ is monotone, then it is an MVE.*

[13] To avoid the usual problems with equilibria in voting games, we assume sequential voting for some fixed sequence of players. See Acemoglu, Egorov, and Sonin (2009) for a solution concept which would refine out unnatural equilibria in voting games with simultaneous voting.

This theorem thus establishes the close connection between MVE and MPE. Essentially, any MVE corresponds to an MPE (for some protocol) and, conversely, any MPE corresponds to an MVE, provided that this MPE induces monotone transitions.

3.4 Comparative statics

In this section, we present general comparative static results. We assume that parameter values are generic. We say that environments E_1 and E_2, defined for the same set of players and set of states, *coincide on* $S' \subset S$, if for each $i \in N$ and for any state $x \in S'$, $u_{E_1,i}(x) = u_{E_2,i}(x)$, $W_{E_1,x} = W_{E_2,x}$, and also $F_{E_1}|_{S'} = F_{E_2}|_{S'}$ (in the sense that for $x, y \in S'$, $y \in F_{E_1}(x) \Leftrightarrow y \in F_{E_2}(x)$).

Our next result shows that in two environments E_1 and E_2 that coincide on a subset of states (and differ arbitrarily on other states), there is a simple way of characterizing the transition mapping of one environment at the steady state of the other. We also say that the MVE is unique on $S' \subset S$ if there exists a unique equilibrium when (transitions are) restricted to the set of states S'. For the results in this section, we assume that there exists a unique MVE (e.g., either set of conditions of Theorem 4 hold).[14]

Theorem 6 ***(General Comparative Statics I)*** *Suppose that environments E_1 and E_2 coincide on $S' = [1, s] \subset S$ and that there is a unique MVE in both environments. For MVE ϕ_1 in E_1, suppose that $\phi_1(x) = x$ for some $x \in S'$. Then for MVE ϕ_2 in E_2 we have $\phi_2(x) \geq x$.*

The theorem says that if x is a steady state (limit state) in environment E_1 and environments E_1 and E_2 coincide on a subset of states $[1, s]$ that includes x, then the MVE in E_2 will either stay at x or induce a transition to a greater state than x. Of course, the two environments can be swapped: if $y \in S'$ is such that $\phi_2(y) = y$, then $\phi_1(y) \geq y$. Moreover, since the ordering of states can be reversed, a similar result applies when $S' = [s, m]$ rather than $[1, s]$.

The intuition for Theorem 6 is instructive. The fact that $\phi_1(x) = x$ implies that in environment E_1, there is no winning coalition wishing to move from x to $y < x$. But when restricted to S', economic payoffs and the distribution of political power are the same in environment E_2 as in E_1, so in environment E_2 there will also be no winning coalition supporting the move to $y < x$. This implies $\phi_2(x) \geq x$. Note, however, that $\phi_2(x) > x$ is possible even though $\phi_1(x) = x$, since

[14] A similar result can be established without uniqueness. For example, one can show that if for some $x \in S'$, for each MVE ϕ_1 in E^1, $\phi_1(x) \geq x$, with at least one MVE ϕ_1 such that $\phi_1(x) = x$, then all MVE ϕ_2 in E^2 satisfy $\phi_2(x) \geq x$. Because both the statements of these results and the proofs are more involved, we focus here on situations in which MVE are unique.

the differences in economic payoffs or distribution of political power in states outside S' may make a move to higher states more attractive for some winning coalition in E_2. Interestingly, since how the two environments differ outside S' is left totally unrestricted, this last possibility can happen even if in environment E_2 payoffs outside S' are *lower* for all players (this could be, for example, because even though all players' payoffs decline outside S', this change also removes some "slippery slope" previously discouraging a winning coalition from moving to some state $z > x$).

The idea of the proof of the theorem also follows from the intuition given in the previous paragraph. To obtain the main idea, let us use the notation $\phi|_{S'}$ to represent the transition function ϕ restricted to the subset of states S'. Now if we had $\phi_2(x) < x$, then $\phi_1|_{S'}$ and $\phi_2|_{S'}$ would be two different mappings, both of which would be MVE on S'. But this would contradict the uniqueness of MVE.

Theorem 6 compares MVE in two distinct environments. In this sense, we can think of it as a comparative static with respect to an unanticipated shock (taking us from one environment to the other). The next corollary states a similar result when there is a stochastic transition from one environment to another.

Corollary 1 *Suppose that $\mathcal{E} = \{E_1, E_2\}$, E_1 and E_2 coincide on $S' = [1, s] \subset S$, and the MVE is unique in both environments. Suppose also that for MVE ϕ_{E_1} in E_1 and some $x \in S'$, $\phi_{E_1}(x) = x$, and this state x is reached before a switch from environment E_1 to E_2 occurs at time t. Then the MVE ϕ_{E_2} in environment E_2 implies that $s_\tau \geq x$ for all $\tau \geq t$.*

Put differently, the corollary states that if steady state x is reached before a shock changes the environment — in a way that only higher states are affected as a result of this change in environment — then the equilibrium after the change can only move society further towards the direction where the shock happened or stay where it was; the equilibrium will never involve moving back to a lower state than x. A straightforward implication is that the only way society can stay in the set of states $[1, x-1]$ is not to leave the set before the shock arrives.

An interesting application of this corollary can be derived when we consider x as a "minimal democratic state"; states to the right of x as further developments of democracy or other refinements; and environment E_2 as representing some sort of threat to democracy. Then the corollary implies that this threat to democracy may disrupt the emergence of this minimal democracy if it arrives early. But if it arrives late, after this minimal democratic state —which thus can be considered as a "democratic threshold" — has already been reached, it would not create a reversal. Interestingly, and perhaps paradoxically, such a threat, if it arrives late, may act as an impetus for additional transitions in a further democratic direction, even though it would have prevented the emergence of this minimum democratic state had it arrived early.

Corollary 1 was formulated under the assumption that stable state x was reached before the shock occurred. Our next result removes this constraint under the assumption that the discount factor is low enough, i.e., that players are sufficiently myopic.

Theorem 7 ***(General Comparative Statics II)*** *Suppose that* $\mathcal{E} = \{E_1, E_2\}$, $0 < \pi(E_1, E_2) < 1$, $\pi(E_2, E_1) = 0$, *and* E_1 *and* E_2 *coincide on* $S' = [1, s] \subset S$. *Then there exists* $\beta_0 > 0$ *such that if* $\beta < \beta_0$, *then in the unique MVE* ϕ, *if the initial state is* $s_0 \in S'$ *such that* $\phi_{E_1}(s_0) \geq s_0$, *then the entire path* $s_0, s_1, s_2, \ldots$ *(induced both under environment* E_1 *and after the switch to* E_2*) is monotone. Moreover, if the shock arrives at time* t, *then for all* $\tau \geq t$, $s_\tau \geq \tilde{s}_\tau$, *where* $\tilde{s}_\tau$ *is the hypothetical path if the shock never arrives.*

In a monotone MVE, equilibrium paths are monotone without shocks. But with shocks, this is no longer true because the arrival of the shock can change the direction of the path. This theorem shows that when the discount factor is sufficiently low and two environments coincide on a subset of states, then the equilibrium path is monotone even with shocks, and equilibrium paths with and without shocks can be ranked.

Under further assumptions on how the shock changes the distribution of political power, we can also derive additional results on the dynamics of equilibrium paths. This is done in the next theorem for the case in which shocks change the set of quasi-median voters — i.e., they change the distribution of political power in a specific way.

Theorem 8 ***(General Comparative Statics III)*** *Suppose that environments* E_1 *and* E_2 *have the same payoffs,* $u_{E_1,i}(x) = u_{E_2,i}(x)$, *that the same transitions are feasible* ($F_{E_1} = F_{E_2}$) *and that* $M_{E_1,x} = M_{E_2,x}$ *for* $x \in [1, s]$ *and* $\min M_{E_1,x} = \min M_{E_2,x}$ *for* $x \in [s+1, m]$. *Suppose also that the MVE* ϕ_1 *in* E_1 *and MVE* ϕ_2 *in* E_2 *are unique on any subset of* $[1, s]$. *Then* $\phi_1(x) = \phi_2(x)$ *for any* $x \in [1, s]$.

This result suggests that if the sets of winning coalition in some states to the right ($x > s$) change such that the sets of quasi-median voters expand further towards the right (for example, because some additional players on the right become additional veto players), then the transition mapping is unaffected for states on the left that are not directly affected by the change (i.e., $x < s$). For instance, applied to the dynamics of democratization, this theorem implies that an absolute monarch's decision of whether to move to a constitutional monarchy is not affected by the power that the poor will be able to secure in this new regime provided that the monarch himself still remains a veto player.

4 Application: Implications of Radical Politics

In this section, we apply our general framework to the study of radical politics, already briefly introduced in Example 1 in the Introduction. We first describe the initial environment, E_1. There is a fixed set of n players $N = \{-l, \ldots, r\}$ (so $n = l + r + 1$), which we interpret as groups of individuals with the same preferences (e.g., ethnicities, economic interests or ideological groupings) that have already solved the within-group collective action problem.

The weight of each group $i \in N$ is denoted by γ_i and represents, for example, the number of individuals within the group and thus its political power. Throughout this exercise, we assume "genericity" of $\{\gamma_i\}$, in the sense that there are no two disjoint combinations of groups with exactly the same weight (see Acemoglu, Egorov and Sonin, 2008, for a discussion of this assumption). Group 0 is chosen such that it contains the median voter. Individuals in group i have preferences (net of repression costs) given

$$w_i(p) = -(p - b_i)^2,$$

where p is the policy choice of society and b_i is the political bliss point of group i. We assume that $\{b_i\}$ is increasing in i, which ensures that preferences satisfy increasing differences (Assumption 2). For example, those with high index can be interpreted as the "rich" or "right-wing" groups that prefer higher levels of the (pro-rich or right-wing) policy.

The set of states is $S = \{-l - r, \ldots, l + r\}$, and so the total number of states is $m = 2l+2r+1 = 2n-1$. States correspond to different combinations of political rights. Political rights of certain groups can be reduced by repression (which is potentially costly as described below). The set of groups that are *not* repressed in state s is denoted by H_s, where $H_s = \{-l, \ldots, r + s\}$ for $s \leq 0$ and $H_s = \{-l + s, \ldots, r\}$ for $s > 0$.[15] Only the groups that are not repressed participate in politics. This implies that in state 0, which corresponds to "democracy" with no

[15] We could allow for the repression of any combination of groups, thus having to consider $2^n - 1$ rather than $2n-1$ states, but choose not to do so to save on notation. Partial repression of some groups could also be allowed, with similar results.

repression, group 0 contains the median voter. In states below 0, some groups with right-wing preferences are repressed, and in the leftmost state $s = -l - r$, only the group $-l$ participates in decision-making (all other groups are repressed). Similarly, in states above 0 some of the left-wing groups are repressed (in rightmost state $s = l + r$ only group r has power). We assume that all transitions across states are feasible.

Policy p and transitions across states are decided by a simple majority of those with political rights (groups that are not repressed). This implies that policy will always be chosen as the political bliss point of the quasi-median voter (given political rights), b_{M_s}. Our assumptions so far (in particular, the genericity of $\{\gamma_i\}$) ensure that M_s contains a single group. The cost of repressing each individual in group j is denoted by C_j and is assumed to be incurred by all players. So stage payoffs are given as

$$\begin{aligned} u_i(s) &= w_i(p) - \sum\nolimits_{j \notin H_s} \gamma_j C_j, \\ &= -\left(b_{M_s} - b_i\right)^2 - \sum\nolimits_{j \notin H_s} \gamma_j C_j. \end{aligned}$$

Finally, we also assume that the radical group $-l$ is smaller than the next group: $\gamma_{-l} < \gamma_{-l+1}$, which implies that radicals can implement their preferred policy only by repressing all of the groups in society.

We model power shifts by introducing h "radical" environments $R_{-l-r}, \ldots, R_{-l-r+h-1}$, each with probability λ_j for $j = 1, \ldots, m$ at each date starting from E_1. Environment R_j is the same as E_1, except that in environment R_j, if the current state is one of $-l - r, \ldots, j$, the radical group, $-l$, acquires the ability to force a transition to any other state (in the process incurring the costs of repression). In particular, the radicals can choose to "grab power" by repressing all other groups and transitioning to state $s = -l - r$. In the context of the Bolshevik Revolution, for example, this corresponds to assuming that in some possible environments (i.e., with some probability), Bolsheviks are able to grab control with Kerensky in power but not necessarily with some further right government. Therefore, in state s, the probability of the radicals having an opportunity to grab power is $\mu_s = \sum_{j=-l-r}^{s} \lambda_j$, which is naturally (weakly) increasing in s.

We also assume that in each period in any of the environments R_j, there is a probability ν of returning to the initial environment, E_1. This is equivalent to a transition to the "final" environment E_f identical to E_1 in terms of payoffs and winning coalitions (but there will be no further possibility of radicals coming to power after that). Clearly, $\nu = 0$ corresponds to a permanent shock, and as ν increases, the expected length of the period during which radicals can dictate transitions declines. Note, however, that if the first time they get the opportunity, radicals grab power permanently, imposing a transition to state $s = -l - r$ (in which they are the dictator), then they will remain in power even after there is a transition to environment E_f.

The next proposition characterizes MVE in an environment in which there is no possibility of a radical takeover of power. This environment can be represented by E_f (since from E_f there is no further transition and thus no possibility of a radical takeover of power), and we use this convention to avoid introducing further notation.

Proposition 1 ***(Equilibria without radicals)*** *Without the possibility of radicals grabbing power (i.e., in environment E_f), there exists a unique MVE represented by $\phi_{E_f} : S \to S$. In this equilibrium:*

1. *Democracy is stable: $\phi_{E_f}(0) = 0$.*

2. *For any costs of repression $\{C_j\}_{j \in N}$, there is never more repression than the initial state: i.e., if $s < 0$ then $\phi_{E_f}(s) \in [-s, 0]$, and if $s > 0$, then $\phi_{E_f}(s) \in [0, s]$.*

3. *Consider repression costs parametrized by k: $C_j = kC_j^*$, where $\left\{C_j^*\right\}$ are positive constants. There exists $k^* > 0$ such that: if $k > k^*$, then $\phi_{E_f}(s) = 0$ for all s, and if $k < k^*$, then $\phi_{E_f}(s) \neq 0$ for some s.*

Without radicals, democracy is stable because the median voter knows that she can choose policies in the future (and can do so without incurring any cost of repression). Nevertheless, other states may also be stable. For instance, starting from a situation in which there is repression of the left, the quasi-median voter in that state may not find it beneficial to reduce repression because this will typically lead to further left policies (relative to the political bliss point of the quasi-median voter). But this type of repression is also limited by the cost of repression which all players, including the quasi-median voter in the initial state, incur. If these costs are sufficiently high, then repression becomes unattractive starting from any state, and democracy becomes the only stable state.

The next proposition shows how political dynamics change when there is a risk of a radical takeover of power.

Proposition 2 ***(Radicals)*** *There exists a unique MVE. Suppose that when the society is at state s, there is a transition to environment R_z (where $z \geq s$) so that radicals can grab power.*

Then, when they have the opportunity, the radicals are more likely to move to state $s = -l - r$ (repressing all other groups) when: (a) they are more radical (meaning their ideal point b_{-l} is lower, i.e., further from 0); (b) they are "weaker" (i.e., z is smaller) in the sense that there is a smaller set of states in which they are able to control power.

This proposition is intuitive. When they have more radical preferences, radicals value the prospect of imposing their political bliss point more and are willing to incur the costs of repression to do so. They are also more likely to do so when they are "weaker" because when z is lower, there is a greater range of states in which they cannot control future transitions, encouraging an immediate transition to $s = -l - r$.

To state our next proposition, we define

$$W_i(s) = u_i(s) + \beta \sum_{z=-l-r}^{-l-r+h-1} \lambda_z V_{R_z,i}(s),$$

which intuitively corresponds to the (counterfactual) expected continuation value of group i when it permanently stays in state $s \in S$ until a shock changes the environment, and from then on follows the MVE play:

Proposition 3 ***(Repression by moderates anticipating radicals)*** *The transition mapping before radicals come to power, ϕ_{E_1}, satisfies the following properties.*

1. *If $s \leq 0$, then $\phi_{E_1}(s) \geq s$.*

2. *If $W_0(0) < W_0(s)$ for some $s > 0$, then there is a state $x \geq 0$ such that $\phi_{E_1}(s) > s$. In other words, there exists some state in which there is an increase in the repression of the left in order to decrease the probability of a radical takeover of power.*

3. *If for all states $y > x \geq 0$, $W_{M_x}(y) < W_{M_x}(x)$, then for all $s \geq 0$, $\phi_{E_1}(s) \leq s$. In other words, repression of the left never increases when the cost of repression increase (e.g., letting $C_j = kC_j^*$, it declines when k increases).*

The first part of the proposition indicates that there is no reason for repression of the right to increase starting from states below $s = 0$; rather, in these states the tendency is to reduce repression. However, the second part shows that if the median voter (in democracy) prefers a more repressive state *when she could counterfactually ensure no further repression unless radicals come to power* (which she cannot do because she is not in control in that state), then there is at least one state from which there will be an increase of repression against the left (which does not necessarily have to be $s = 0$). An important implication of this result is that even if there are "slippery slope" considerations, these are not sufficient to prevent all repression. The third part of the proposition provides a sufficient condition for the opposite result.

The first part implies that anticipation of a radical takeover of power leads to (weakly) greater repression, at least starting from a sufficiently "democratic" state. In particular, in some states $s > 0$, there will necessarily be lower repression after the threat of radicals disappears.

Proposition 4 ***(Stability of democracy without a threat of radical)*** *Suppose that full democracy $s = 0$ does not allow for radicals coming to power (i.e., $\mu_s = 0$). Then $s = 0$ is stable in all environments, and any state $s > 0$ will lead to (weakly) less repression, in the sense that $\phi_{E_1}(s) \in [0, s]$ for $s > 0$.*

Proposition 4 shows that if democracy is resilient against radicals' power grabs, then it is stable regardless of the possibility of radicals taking over power in other, less democratic states.

The next proposition is an application of our general comparative static results given in Theorem 6.

Proposition 5 ***(Comparative statics of repression)*** *Suppose that there is a state $s \geq 0$ (i.e., democracy or some state favoring the right), which is stable in E_1 for some set of probabilities $\{\mu_j\}$. Consider a change from $\{\mu_j\}$ to $\left\{\mu'_j\right\}$ such that $\mu'_j = \mu_j$ for $j \geq s$. Then there will be (weakly) less repression of the left after the change, i.e., $\phi'_{E_1}(s) \geq \phi_{E_1}(s) = s$.*

The intuition is the same as Theorem 6: if the probabilities of a radical takeover of power change, but only in states that already had repression against the left, and we are in a stable state without repression against the right, then this can only reduce repression. If there is now a lower likelihood of a radical grab of power, then this favors less repression. But, paradoxically, even if there is a higher likelihood of such a grab, because of reduced "slippery slope" considerations, there may be less repression.

The next result compares the transition in anticipation of radicals (environment E_1) and in the case where radicals are gone — or, equivalently, if they are impossible (environment E_f). An implication of this result is a particular type of history dependence in steady-state regime.

Proposition 6 ***(Role of radicals in history)***

1. *If for $s \geq 0$, $\phi_{E_1}(s) = 0$, then $\phi_{E_f}(s) = 0$.*

2. *Suppose the society was in a stable state $x \geq 0$ (in environment E_1) before the radical came to power. Then the limit state (as $t \to \infty$), after the radicals come and possibly go (n environment E_f), will be some $y \leq x$.*

The first part of this proposition shows that the society is at least as likely to cease and any repression and fully democratize once radicals are gone as it is when the arrival of radicals is possible (conversely, if $\phi_{E_f}(s) > 0$, then necessarily $\phi_{E_1}(s) > 0$).[16] Intuitively, initially democratization increases the chance of a radical grab of power, and hence radicals democratization is (weakly) more likely after the radicals are gone. The second part has a related logic and establishes a type of history dependence: the arrival and possible departure of radicals will never lead to more repression of the left than the initial situation and may lead to less repression. This may happen in two ways. First, and less interestingly, the radicals may lock in power forever. Second, when they do not or cannot do that, because the threat of radicals has disappeared, there will be less repression in the stable state.

In the next result, we apply Theorem 8 to show that all the results come from radicals grabbing power rather than just becoming influential enough to become veto players. For this proposition, let us expand our environment to allow for veto players (instead of all decisions being made by majoritarian voting among groups with political rights).

Proposition 7 ***(Radicals as veto players)*** *If shocks make radicals veto players while preserving democratic decision-making, then mapping $\phi(s)|_{s\geq 0}$ is the same as in the benchmark case where the initial environment is stable.*

The intuitionis simple: the current quasi-median voter fears a radical power grab and subsequent dictatorship. If the risk is that the radicals will just become veto players, this is not sufficient to induce repression against the left.

Our last result deals with strategic complementarity in repressions. To state this result, consider a change in the costs of repression so that it becomes cheaper for radicals to repress right-wing groups. In particular, the state payoff function of radicals changes to

$$u_{-l}(s) = -\left(b_{M_s} - b_{-l}\right)^2 - \rho \sum\nolimits_{j \notin H_s} \gamma_j C_j$$

[16] It is also straightforward to construct an example where $\phi_{E_f}(s) = 0$ but $\phi_{E_1}(s) > 0$ (and even $\phi_{E_1}(s) > s$).

for $s < 0$ and $\rho \in [0, 1]$. Clearly, $\rho = 1$ corresponds to our baseline environment, and a decrease in ρ implies that radicals can repress right-wing groups with less cost to themselves. Then:

Proposition 8 ***(Strategic Complementarity)*** *Suppose that $\lambda_z = 0$ for all $z > 0$ (meaning that radicals can only seize power if they are not currently repressed). Consider a change in the radicals' repression costs to $\rho' < \rho$ and denote the MVE before and after the change by ϕ and ϕ' respectively. Then if $\phi_{E_1}(s) > s$ for some $s \geq 0$, $\phi'_{E_1}(s) > s$.*

Put differently, the proposition implies that if $\phi_{E_1}(0) > 0$, then $\phi'_{E_1}(0) > 0$, so that repression of the radicals is more likely when they themselves have lower costs of repressing other groups. At the root of this result is a strategic complementarity in repression: anticipating greater repression by radicals in future radical environments, the current political system now becomes more willing to repress the radicals. One interesting implication of this result is that differences in repression of different ends of the political spectrum across societies may result from small differences in (institutional or social) costs of repression rather than a "culture of repression" in some countries. Thus, the brutal repression of first left- and then right-wing groups in early 20th-century Russia, contrasted with a lack of such systematic repression in Britain may not just be a reflection of a Russian culture of repression, but a game-theoretic consequence of the anticipation of different patterns of repression in different political states in Russia.

5 Extensions

In this section, we first provide (simple and relatively mild) conditions under which all MVE are monotone. This justifies our focus on monotone MVE throughout the rest of the paper. We then relate our paper in more detail to Roberts (1999) discussed already in the Introduction. We also discuss how our results will be different with infinitely many shocks. Finally, we show how our framework can be extended to economies with a continuum of states and/or players.

5.1 Monotone vs nonmonotone MVE

So far, we focused on monotone MVE. In many interesting cases this is without loss of generatlity, as the following theorem establishes.

Theorem 9 ***(Monotonicity of MVE)*** *Under either of the following conditions, all MVE are generically monotone:*

1. *In all environments, the sets of quasi-median voters in two different states have either zero or exactly one player in common: for all $E \in \mathcal{E}, x, y \in S : x \neq y \Rightarrow |M_{E,x} \cap M_{E,y}| \leq 1$.*

2. *In all environments, only one-step transitions are possible.*

The first part of the theorem covers, among others, situations where the sets of quasi-median voters are singletons in all states. This implies that whenever there is a dictator in each state (which may be the same for several states), or there is majority voting among sets of odd numbers of players, any MVE is monotone, and thus all results in the paper are applicable to all MVE —rather than the monotone subset of MVE. The second part shows that if only one-step transitions, i.e., transitions to adjacent states, are possible, then again any MVE is monotone. This means that our focus on monotone MVE is with little loss of generality for many interesting and relevant cases.

Note also that the conditions in Theorem 9 are weaker than those in Theorem 2 and 4. Consequently, when these latter theorems ensure the uniqueness of a monotone MVE, they also imply that the MVE is in fact unique.

The next example shows that both conditions in Theorem 9 cannot be simultaneously dispensed with.

Example 3 There are three states A, B, C, and two players 1 and 2. The decision-making rule is unanimity in all states, and all transitions are possible. Payoffs are given by

id	A	B	C
1	30	50	40
2	10	40	50

Suppose β is relatively close to 1, e.g., $\beta = 0.9$. This situation does not satisfy either set of conditions of Theorem 9. It is straightforward to verify that there is a nonmonotone MVE $\phi(A) = \phi(C) = C$, $\phi(B) = B$. (There is also a monotone equilibrium with $\phi(A) = \phi(B) = B$, $\phi(C) = C$.)

The next example shows that monotonicity may fail non-generically even when the conditions of Theorem 9 are satisfied.

Example 4 There are two states A and B and two players 1 and 2. Player 1 is the dictator in both stattes. Payoffs are given by

id	A	B
1	20	20
2	15	25

Take any discount factor β, e.g., $\beta = 0.5$, and any protocol. Then there exists a nonmonotone (in fact, cyclic) MVE ϕ given by $\phi(A) = B$ and $\phi(B) = A$. However, any perturbation of the payoffs of player 1 removes this nonmonotonic equilibrium.

Our last result in this section shows that even if nonmonotone MVE exist, they will still induce "monotone paths". We say that mapping $\phi = \{\phi_E\}_{E \in \mathcal{E}}$ induces *monotone paths* if for any $E \in \mathcal{E}$ and $x \in S$, $\phi(x) \geq x$ implies $\phi_E^2(x) \geq \phi_E(x)$.

In other words, all equilibrium paths that this mapping generates, as long as the environment does not change, are weakly monotone. We have the following result:

Theorem 10 ***(Monotone Paths)*** *Any MVE ϕ (not necessarily monotone) generically induces monotone paths.*

5.2 Relationship to Roberts's model

As discussed in the Introduction, our paper is most closely related to Roberts (1999). Our notion of MVE extends that of Roberts, who also looks at a dynamic equilibrium in an environment that satisfies single-crossing type restrictions. More specifically, in Roberts's model, the society consists of n players, and there are n possible states $s_k = \{1, \ldots, k\}$, $1 \leq k \leq n$. Each state s_k describes the situation where players $\{1, \ldots, k\}$ are members of the club, while others are not. There is the following condition on payoffs:

$$\text{for all } l > k \text{ and } j > i, \;\; u_j(s_l) - u_j(s_k) > u_i(s_l) - u_i(s_k),$$

which is the same as the strict increasing differences condition we imposed above (Definition 1).

Roberts (1999) focuses on deterministic environments with majoritarian voting among club members. He then looks at a notion of Markov Voting Equilibrium (defined as an equilibrium path where there is a transition to a new club whenever there is an absolute majority in favor of it) and a median voter rule (defined as an equilibrium path where at each point the current median voter chooses the transition for the next step). Roberts proves existence for mixed-strategy equilibria for each of the voting rules; they define the same set of clubs that are stable under these rules.

Roberts's notion of Markov Voting Equilibrium is also a special case of ours. When our notion is specialized to majoritarian voting, the two differ only in their treatment of situations with "clubs" with even numbers of members.

Overall, our setup and results generalize, extend and strengthen Robert's in several dimensions. First, Roberts focuses on the deterministic and stationary environment, whereas we allow for nonstationary elements and rich stochastic shocks. Second, we allow for fairly general distributions of political power across states, which is crucial for our focus, while Roberts assumes majority rule for every club. Third, we prove existence of pure strategy equilibria and provide conditions for uniqueness (results that do not have equivalents in Roberts). Fourth, we provide a general characterization of the structure of MVE, which in turn paves the way for our general comparative static results (again results that have no equivalents and Roberts). Fifth, we show the relationship between this equilibrium concept and MPE of a fully specified dynamic game. Finally, we show how our framework can be applied to a political economy problem, providing new and interesting insights in this instance.

5.3 Infinitely many shocks

Suppose that there is a finite set of environments $\mathcal{E}$, but we relax Assumption 1, so that there can be an infinite number of shocks . In this case, MVE (as defined in Definition 3) may fail to exist. Example 8 in the Appendix illustrates this possibility.

The reason why MVE may fail to exist is as follows. Take some set of mappings $\phi = \{\phi_E\}$ and assume that they define transitions from period T onwards (for some large T). Using the same technique as in Section 3.2, we can show existence of a mapping $\phi^{[T-1]} = \left\{\phi_E^{[T-1]}\right\}$ which would determine transitions in period $T-1$; then we can do the same for period $T-2$, etc. The problem is that these mappings may be different for different periods, whereas the natural Markovian property would be to impose that they should be the same. Therefore, with infinitely many shocks, there exists a pure strategy equilibrium without this latter Markovian requirement, but if we would also like to insist on this Markovian requirement, one has to work with mixed strategies.[17]

5.4 Continuous spaces

In this subsection, we show how our results can be extended to economies with a continuum of states and/or a continuum of players.

Suppose that the set of states is $S = [s_l, s_h]$, and the set of players is given by $N = [i_l, i_h]$. (The construction and reasoning below are easily extendable to the case where the are a finite

[17] Details are available from the authors upon request.

number of players but a continuum of states, or vice versa.) We assume that each player has a utility function $u_i(s) : S \to \mathbb{R}$, which is continuous as a function of $(i,s) \in N \times S$ and satisfies strict increasing differences: for all $i > j$, $x > y$,

$$u_i(x) - u_i(y) > u_j(x) - u_j(y).$$

The mapping F, which describes feasible transitions, is assumed to be upper-hemicontinuous on S and to satisfy Assumption 5. Finally, for each state s there is a set of winning coalitions W_s, which are assumed to satisfy Assumption 3. As before, for each state s, we have a non-empty set of quasi-median voters M_s (which may nevertheless be a singleton). We make the following monotonicity of quasi-median voters assumption: functions $\inf M_s$ and $\sup M_s$ are continuous and increasing functions of s.

For simplicity, let us focus on the case without shocks and on monotone transition functions $\phi : S \to S$ (this function may be discontinuous). MVE is defined as in Definition 3. The following result establishing the existence of MVE.

Theorem 11 ***(Existence in Continuous Spaces)*** *With a continuum of states and/or players, there exists a MVE ϕ. Moreover, take any sequence of sets of states $S_1 \subset S_2 \subset \cdots$ and any sequence of players $N_1 \subset N_2 \subset \cdots$ such that $\bigcup_{j=1}^{\infty} S_j$ is dense in S and $\bigcup_{j=1}^{\infty} N_j$ is dense in N. Consider any sequence of monotone functions $\left\{\phi_j : S_j \to S_j\right\}_{j=1}^{\infty}$ which are MVE (not necessarily unique) in the environment*

$$E_j = \left(N, S, \beta, \{u_i(s)\}_{i \in N_j}^{s \in S_j}, \{W_s\}_{s \in S_j}, \{F_j(s)\}_{s \in S_j}\right).$$

Existence of such MVE is guaranteed by Theorem 1, as all assumptions are satisfied. Then there is a subsequence $\{j_k\}_{k=1}^{\infty}$ such that $\left\{\phi_{j_k}\right\}_{k=1}^{\infty}$ converges pointwise on $\bigcup_{j=1}^{\infty} S_j$, to some MVE $\phi : S \to S$.

This result therefore shows that an MVE exists and is extended environment and may be characterized as a limit of equilibria for finite sets of states and players. The idea of the proof is simple. Take an increasing sequence of sets of states , $S_1 \subset S_2 \subset \cdots$ and an increasing sequence of sets of players $N_1 \subset N_2 \subset \cdots$ such that $S_\infty = \bigcup_{j=1}^{\infty} S_j$ is dense in S and $N_\infty = \bigcup_{j=1}^{\infty} N_j$ is dense in N. For each S_j, take MVE ϕ_j. We know that ϕ_i is a monotone function on S_i. Let us extend it to a monotone (not necessarily continuous) function on S which we denote by $\tilde{\phi}_i$ for each i. Since S_∞ and N_∞ are countable, there is a subsequence ϕ_{j_k} which converges to some

$\phi : S_\infty \to S_\infty$ pointwise. We then extend it to a function $\phi : S \to S$ by demanding that ϕ is either left-continuous or right-continuous at any point (in the Appendix, we show that we can do that so that the continuation values are either left-continuous or right-continuous as well). Then this continuity of continuation values will ensure that ϕ is an MVE.

6 Conclusion

This paper has provided a general framework for the analysis of dynamic political economy problems, including democratization, extension of political rights or repression of different groups. The distinguishing feature of our approach is that it enables the analysis of non-stationary, stochastic environments (e.g., allowing for anticipated and unanticipated shocks changing the distribution of political power and economic payoffs) under fairly rich heterogeneity and general political or economic conflict across groups.

We assume that the payoffs are defined either directly on states or can be derived from states, which represent economic and political institutions. For example, different distribution of property rights or adoption of policies favoring one vs. another group correspond to different states. Importantly, states also differ in their distribution of political power: as states change, different groups become politically pivotal (and in equilibrium different coalitions may form). Our notion of equilibrium is Markov Voting Equilibrium, which requires that economic and political changes — transitions across states — should take place if there exists a subset of players with the power to implement such changes and who will obtain higher expected discounted utility by doing so.

We assume that both states and players are "ordered": e.g., states go from more right-wing to more left-wing (or less to more democratic) and players are ordered according to their ideology or income level. Our most substantive assumptions are that, given these orders, stage payoffs satisfy a "single crossing" (increasing differences) type assumption, and the distribution of political power also shifts in the same direction of economic preferences (e.g., more right-wing individuals gain relatively more from moving towards right-wing states than do left-wing individuals, and their political power does not decrease if there is a transition towards such a right-wing state).

Under these assumptions, we prove the existence of a pure-strategy equilibrium, provide conditions for its uniqueness, and show that a limit state always exists (though it generally depends on the order and exact timing of shocks). We also provide a number of comparative static results that apply at this level of generality. For example, if there is a change from one environment to another (with different economic payoffs and distribution of political power) but the two environments coincide up to a certain state s' and before the change the steady state of equilibrium was at some state $x \leq s'$, then the new steady state after the change in environment can be no smaller than x.

We then use this framework to study the dynamics of repression in the presence of radical groups that can stochastically grab power depending on the distribution of political rights in society. We characterize the conditions under which the presence of radicals leads to greater repression (of less radical groups), show a type of path dependence in politics resulting from radicals coming to power, and identify a novel strategic complementarity in repression.

There are several extensions of this framework that would be useful. These include: a generalization of the results to an infinite number of shocks (our analysis was simplified by assuming that there are at most a finite number of transitions); greater individual-level heterogeneity, which can change over time (e.g., a type of "social mobility"); and most importantly extensions of the results to environments in which heterogeneity cannot be captured by a single dimensional order. There are also several additional applications of our framework to problems in political economy, organizational economics and public economics, which can be investigated in future work.

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Appendix

Proofs

Proof of Lemma 1. "If": Suppose $M_s \subset P$, so for each $i \in M_s$, $w_i(y) > w_i(x)$. Consider two cases. If $y > x$, then increasing differences implies that $w_j(y) > w_j(x)$ for all $j \geq \min M_s$. On the other hand, $[\min M_s, n]$ is a winning coalition (if not, $i = M_s - 1$ would be a QMV by definition, but such $i \notin M_s$). If $y < x$, then, similarly, $w_j(y) > w_j(x)$ for all $j \leq \max M_s$, which is a winning coalition for similar reasons. In either case, P contains a subset (either $[\min M_s, n]$ or $[1, \max M_s]$) which is a winning coalition, and thus $P \in W_s$.

"Only if": Suppose $P \in W_s$. Consider the case $y > x$. Let $i = \min P$; then increasing differences implies that for all $j \geq i$, $w_j(y) > w_j(x)$. This means that $P = [i, n]$, and is thus a connected coalition. Since P is winning, we must have $i \leq j \leq n$ for any $j \in M_s$ by definition of M_s, and therefore $M_s \subset P$. The case where $y < x$ is similar, so $M_s \subset P$.

The proofs for relations $\geq, <, \leq$ are similar and are omitted. ∎

Proof of Lemma 2. Part 1. Take $y > x$ and any $i \in N$. We have:

$$\begin{aligned} V_i^{\phi}(y) - V_i^{\phi}(x) &= u_i(y) + \sum\nolimits_{k=1}^{\infty} \beta^k u_i\left(\phi^k(y)\right) - u_i(x) - \sum\nolimits_{k=1}^{\infty} \beta^k u_i\left(\phi^k(x)\right) \\ &= (u_i(y) - u_i(x)) + \sum\nolimits_{k=1}^{\infty} \beta^k \left(u_i\left(\phi^k(y)\right) - u_i\left(\phi^k(x)\right)\right). \end{aligned}$$

The first term is (weakly) increasing in i if $\{u_i(s)\}_{i \in N}^{s \in S}$ satisfies increasing differences, and the second is (weakly) increasing in i as $\phi^k(y) \geq \phi^k(x)$ for $k \geq 1$ due to monotonicity of ϕ. Consequently, (4) is (weakly) increasing in i.

Part 2. If ϕ is monotone, then Part 1 applies. Otherwise, for some $x < y$ we have $\phi(x) > \phi(y)$, and this means that $y = x + 1$; there may be one or more such pairs. Notice that for such x and y, we have $\phi(x) = y$ and $\phi(y) = x$. Consider

$$\begin{aligned} V_i^{\phi}(y) - V_i^{\phi}(x) &= \left(u_i(y) + \sum\nolimits_{k=1}^{\infty} \beta^{2k-1} u_i(x) + \sum\nolimits_{k=1}^{\infty} \beta^{2k} u_i(y)\right) \\ &\quad - \left(u_i(x) + \sum\nolimits_{k=1}^{\infty} \beta^{2k-1} u_i(y) + \sum\nolimits_{k=1}^{\infty} \beta^{2k} u_i(x)\right) \\ &= \frac{1}{1+\beta}(u_i(y) - u_i(x)); \end{aligned}$$

this is (weakly) increasing in i.

Let us now modify stage payoffs and define

$$\tilde{u}_i(x) = \begin{cases} u_i(x) & \text{if } \phi(x) = x \text{ or } \phi^2(x) \neq x; \\ (1-\beta) V_i(x) & \text{if } \phi(x) \neq x = \phi^2(x). \end{cases}$$

Consider mapping $\tilde{\phi}$ given by

$$\tilde{\phi}(s) = \begin{cases} \phi(x) & \text{if } \phi(x) = x \text{ or } \phi^2(x) \neq x; \\ x & \text{if } \phi(x) \neq x = \phi^2(x). \end{cases}$$

This $\tilde{\phi}$ is monotone and $\{\tilde{u}_i(x)\}_{i \in N}^{x \in S}$ satisfies increasing differences. By Part 1, the continuation values $\left\{\tilde{V}_i^{\tilde{\phi}}(x)\right\}_{i \in N}^{x \in S}$ computed for $\tilde{\phi}$ and $\{\tilde{u}_i(x)\}_{i \in N}^{x \in S}$ using (4) satisfy increasing differences as well. But by construction, $\tilde{V}_i^{\tilde{\phi}}(x) = V_i^{\phi}(x)$ for each i and s, and thus $\left\{V_i^{\phi}(x)\right\}_{i \in N}^{x \in S}$ satisfies increasing differences. ■

Proof of Lemma 3. Suppose, to obtain a contradiction, that for each $x, y \in S$ such that $y \in F(x)$ and (5) holds, ϕ' given by (6) is not monotone.

Take $x, y \in S$ such that $|y - \phi(x)|$ is minimal among all such pairs $x, y \in S$ (informally, we consider the shortest deviation). By our assertion, ϕ' is not monotone. Since ϕ is monotone and ϕ and ϕ' differ by the value at x only, there are two possibilities: either for some $z < x$, $y = \phi'(x) < \phi(z) \leq \phi(x)$ or for some $z > x$, $\phi(x) \leq \phi(z) < \phi'(x) = y$. Assume the former (the latter case may be considered similarly). Let s be defined by

$$s = \min(z \in S : \phi(z) > y);$$

in the case under consideration, the set of such z is nonempty (e.g., x is its member, and z found earlier is one as well), and hence state s is well-defined. We have $s < x$; since ϕ is monotone, $\phi(s) \leq \phi(x)$.

Notice that a deviation in state s from $\phi(s)$ to y is monotone: indeed, there is no state $\tilde{z}$ such that $\tilde{z} < s$ and $y < \phi(\tilde{z}) \leq \phi(s)$ by construction of s, and there is no state $\tilde{z} > s$ such that $\phi(s) \leq \phi(\tilde{z}) < y$ as this would contradict $\phi(s) > y$. Moreover, it is feasible, so $y \in F(s)$: this is automatically true if $y = s$; if $y > s$, this follows from $s < y < \phi(s)$; and if $y < s$, this follows from $y = \phi'(x)$ and $y < s \leq x$. By assertion, this deviation cannot be profitable, i.e., $V^{\phi}(y) \not>_s V^{\phi}(\phi(s))$. By Lemma 2, since $y < \phi(s)$, $V^{\phi}_{\max M_s}(y) \leq V^{\phi}_{\max M_s}(\phi(s))$. Since $s < x$, Assumption 4 implies (for $i = \max M_x$) $V_i^{\phi}(y) \leq V_i^{\phi}(\phi(s))$.

On the other hand, (5) implies $V_i^{\phi}(y) > V_i^{\phi}(\phi(x))$. We therefore have

$$V_i^{\phi}(\phi(s)) \geq V_i^{\phi}(y) > V_i^{\phi}(\phi(x)) \tag{A1}$$

and thus, by Lemma 2, since $\phi(s) < \phi(x)$ (we know $\phi(s) \leq \phi(x)$, but $\phi(s) = \phi(x)$ would contradict (A1)),

$$V^{\phi}(\phi(s)) >_x V^{\phi}(\phi(x)).$$

Notice, however, that $y < \phi(s) < \phi(x)$ implies that $|\phi(s) - \phi(x)| < |y - \phi(x)|$. This contradicts the choice of y such that $|y - \phi(x)|$ is minimal among pairs $x, y \in S$ such that $y \in F(x)$ and (5) is satisfied. This contradiction proves that our initial assertion was wrong, and this proves the lemma. ■

Proof of Lemma 4. We show first that if [1] is the case, then [2] and [3] are not satisfied. We then show that if [1] does not hold, then either [2] or [3] are satisfied, and finish the proof by showing that [2] and [3] are mutually exclusive.

First, suppose, to obtain a contradiction, that both [1] and [2] hold. Then [2] implies that for some $z \in [a+1, \phi(a+1)]$ such that $z \in F(a)$, $V^{\phi}(z) >_a V^{\phi}(\phi(a))$, but this contradicts that ϕ is MVE, so [1] cannot hold. We can similarly prove that if [1] holds, then [3] is not satisfied.

Second, suppose that [1] does not hold. Notice that for any $x \in S$, $\phi(x) \in F(x)$ and $V^{\phi}(\phi(x)) \geq_x V^{\phi}(x)$, because these properties hold for ϕ_1 if $x \in [1, a]$ and for ϕ_2 if $x \in [a+1, m]$. Consequently, if ϕ is not MVE, then it is because the (core) condition in Definition 3 is violated. Lemma 3 then implies existence of a monotone deviation, i.e., $x, y \in S$ such that $y \in F(x)$ and $V^{\phi}(y) >_x V^{\phi}(\phi(x))$. Since ϕ_1 and ϕ_2 are MVE on their respective domains, we must have that either $x \in [1, a]$ and $y \in [a+1, m]$ or $x \in [a+1, m]$ and $y \in [1, m]$. Assume the former; since the deviation is monotone, we must have $x = a$ and $a + 1 \leq y \leq \phi(a+1)$. Hence, we have $V^{\phi}(y) >_a V^{\phi}(\phi(a))$, and this shows that [2] holds. If we assumed the latter, we would similarly get that [3] holds. Hence, if [1] does not hold, then either [2] or [3] does.

Third, suppose that both [2] and [3] hold. Let

$$\begin{aligned} x &\in \arg \max_{z \in [\phi(a), \phi(a+1)] \cap F(a)} V^{\phi}_{\min M_a}(z), \\ y &\in \arg \max_{z \in [\phi(a), \phi(a+1)] \cap F(a+1)} V^{\phi}_{\max M_{a+1}}(z); \end{aligned}$$

then $x \geq a + 1 > a \geq y$. By construction, $V^{\phi}_{\min M_a}(x) > V^{\phi}_{\min M_a}(y)$ and $V^{\phi}_{\max M_{a+1}}(y) > V^{\phi}_{\max M_{a+1}}(x)$ (the inequalities are strict because they are strict in [2] and [3]). But this violates the increasing differences that $\left\{V^{\phi}_i(s)\right\}^{s \in S}_{i \in N}$ satisfies as ϕ is monotone (indeed, $\min M_a \leq \max M_{a+1}$ by Assumption 4). This contradiction proves that [2] and [3] are mutually exclusive, which completes the proof. ■

For the proof of Theorem 1, the following auxiliary result (which is itself a corollary of Lemma 4) is helpful.

Lemma 5 ***(Extension of Equilibrium)*** *Let $S_1 = [1, m-1]$ and $S_2 = \{m\}$. Suppose that $\phi : S_1 \to S_1$ is a monotone MVE, and that $F(m) \neq \{m\}$. Let*

$$a = \max \left(\arg \max_{b \in [\phi(m-1), m-1] \cap F(m)} V^{\phi}_{\max M_m}(b) \right). \tag{A2}$$

If

$$V^{\phi}(a) >_m u(m) / (1-\beta), \tag{A3}$$

then mapping $\phi' : S \to S$ defined by

$$\phi'(s) = \begin{cases} \phi(s) & \text{if } s < m \\ a & \text{if } s = m \end{cases}$$

is MVE. A similar statement, mutatis mutandis, applies for $S_1 = \{1\}$ and $S_2 = [2, m]$.

Proof of Lemma 5. Mapping ϕ' satisfies property 1 of Definition 3 by construction. Let us show that it satisfies property 2. Suppose, to obtain a contradiction, that this is not the case. By Lemma 3, there are states $x, y \in S$ such that

$$V^{\phi'}(y) >_x V^{\phi'}(\phi'(x)), \tag{A4}$$

and this deviation is monotone. Suppose first that $x < m$, then $y \leq \phi(m) = a \leq m-1$. For any $z \leq m-1$, $(\phi')^k(z) = \phi^k(z)$ for all $k \geq 0$, and thus $V^{\phi'}(z) = V^{\phi}(z)$; therefore, $V^{\phi}(y) >_x V^{\phi}(\phi(x))$. However, this would contradict that ϕ is a MVE on S_1. Consequently, $x = m$. If $y < m$, then (A4) implies, given $a = \phi'(m)$,

$$V^{\phi}(y) >_m V^{\phi}(a). \tag{A5}$$

Since the deviation is monotone, $y \in [\phi(m-1), m-1]$, but then (A5) contradicts the choice of a in (A2). This implies that $x = y = m$, so (A4) may be rewritten as

$$V^{\phi'}(m) >_m V^{\phi}(a). \tag{A6}$$

But since

$$V^{\phi'}(m) = u(m) + \beta V^{\phi}(a), \tag{A7}$$

(A6) implies

$$u(m) >_m (1-\beta) V^{\phi}(a).$$

This, however, contradicts (A3), which proves that ϕ' satisfies property 2 of Definition 3.

To prove that ϕ' is MVE, we need to establish that it satisfies property 3 of Definition 3, i.e.,

$$V^{\phi'}\left(\phi'\left(x\right)\right) \geq_x V^{\phi'}\left(x\right) \tag{A8}$$

for each $x \in S'$. If $x \in S$ (i.e., $x < m$), then $\left(\phi'\right)^k\left(x\right) = \phi^k\left(x\right)$ for any $k \geq 0$, so (A8) is equivalent to $V^{\phi}\left(\phi\left(x\right)\right) \geq_x V^{\phi}\left(x\right)$, which is true for $x < m$, because ϕ is MVE on S. It remains to prove that (A8) is satisfied for $x = m$. In this case, (A8) may be rewritten as

$$V^{\phi}\left(a\right) \geq_m V^{\phi'}\left(m\right). \tag{A9}$$

Taking (A7) into account, (A9) is equivalent to $\left(1-\beta\right)V^{\phi}\left(a\right) \geq_m u\left(m\right)$,which is true, provided that (A3) is satisfied. We have thus proved that ϕ' is MVE on S', which completes the proof. ■

Proof of Theorem 1. We prove this result by induction by the number of states. For any set X, let Φ^X be the set of monotone MVE, so we have to prove that $\Phi^X \neq \varnothing$.

Base: If $m = 1$, then $\phi : S \to S$ given by $\phi\left(1\right) = 1$ is MVE for trivial reasons.

Induction Step: Suppose that if $|S| < m$, then MVE exists. Let us prove this if $|S| = m$. Consider the set $A = [1, m-1]$, and for each $a \in A$, consider two monotone MVE $\phi_1^a : [1, a] \to [1, a]$ and $\phi_2^a : [a+1, m] \to [a+1, m]$. Without loss of generality, we may assume that

$$\begin{aligned}
\phi_1^a &\in \arg \max_{\phi \in \Phi^{[1,a]}, z \in [\phi(a), a] \cap F(a+1)} V^{\phi}_{\max M_{a+1}}\left(z\right), \\
\phi_2^a &\in \arg \max_{\phi \in \Phi^{[a+1,m]}, z \in [a+1, \phi(a+1)] \cap F(a)} V^{\phi}_{\min M_a}\left(z\right)
\end{aligned}$$

(whenever $[\phi\left(a\right), a] \cap F\left(a+1\right) = \varnothing$ or $[a+1, \phi\left(a+1\right)] \cap F\left(a\right)$ are empty, we pick any ϕ_1^a or ϕ_2^a, respectively). For each $a \in A$, define $\phi^a : S \to S$ by

$$\phi^a\left(s\right) = \begin{cases} \phi_1^a\left(s\right) & \text{if } s \leq a \\ \phi_2^a\left(s\right) & \text{if } s > a \end{cases}.$$

Let us define function $f : A \to \{1, 2, 3\}$ as follows. By Lemma 4, for every split $S = [1, a] \cup [a+1, m]$ given by $a \in A$ and for MVE ϕ_1^a and ϕ_2^a, exactly one of three properties hold; let $f\left(a\right)$ be the number of the property. Then, clearly, if for some $a \in A$, $f\left(a\right) = 1$, then ϕ^a is a monotone MVE by construction of function f.

Now let us consider the case where for every $a \in A$, $f\left(a\right) \in \{2, 3\}$. We have the following possibilities.

First, suppose that $f\left(1\right) = 2$. This means that (since $\phi_1^a\left(1\right) = 1$ for $a = 1$)

$$\arg \max_{z \in [1, \phi(2)] \cap F(1)} V^{\phi^1}_{\min M_1}\left(z\right) \subset \left[2, \phi^1\left(2\right)\right]. \tag{A10}$$

Let

$$b \in \arg \max_{z \in [2, \phi(2)] \cap F(1)} V^{\phi^1}_{\min M_1}(z) \tag{A11}$$

and define $\phi' : S \to S$ by

$$\phi'(s) = \begin{cases} b & \text{if } s = 1 \\ \phi^1(s) & \text{if } s > 1 \end{cases}; \tag{A12}$$

let us prove that ϕ' is a MVE. Notice that (A10) and (A11) imply

$$V^{\phi^1}_{\min M_1}(b) > V^{\phi^1}_{\min M_1}(1).$$

By Lemma 2, since $b > 1$,

$$V^{\phi^1}(b) >_1 V^{\phi^1}(1). \tag{A13}$$

Notice, however, that

$$V^{\phi^1}(1) = u(1) / (1 - \beta),$$

and also $V^{\phi^1}(b) = V^{\phi^1_2}(b)$; therefore, (A13) may be rewritten as

$$V^{\phi^1_2}(b) >_1 u(1) / (1 - \beta).$$

By Lemma 5, $\phi' : S \to S$ defined by (A12), is a MVE.

Second, suppose that $f(m-1) = 3$. In this case, using the first part of Lemma 5, we can prove that there is a MVE similarly to the previous case.

Finally, suppose that $f(1) = 3$ and $f(m-1) = 2$ (this already implies $m \geq 3$), then there is $a \in [2, m-1]$ such that $f(a-1) = 3$ and $f(a) = 2$. Define, for $s \in S \setminus \{a\}$ and $i \in N$,

$$V^*_i(s) = \begin{cases} V^{\phi^{a-1}_1}_i(s) & \text{if } s < a \\ V^{\phi^a_2}_i(s) & \text{if } s > a \end{cases}.$$

Let us first prove that there exists $b \in \left(\left[\phi^{a-1}_1(a-1), a-1\right] \cup [a+1, \phi^a_2(a+1)]\right) \cap F(a)$ such that

$$V^*(b) >_a u(a) / (1 - \beta), \tag{A14}$$

and let B be the set of such b (so $B \subset \left(\left[\phi^{a-1}_1(a-1), a-1\right] \cup [a+1, \phi^a_2(a+1)]\right) \cap F(a)$). Indeed, since $f(a-1) = 3$,

$$\arg \max_{z \in \left[\phi^{a-1}(a-1), \phi^{a-1}(a)\right] \cap F(a)} V^{\phi^{a-1}}_{\max M_a}(z) \subset \left[\phi^{a-1}(a-1), a-1\right]. \tag{A15}$$

Let

$$b \in \arg \max_{z \in \left[\phi^{a-1}(a-1), a-1\right] \cap F(a)} \left(V^{\phi^{a-1}}_{\max M_a}(z)\right), \tag{A16}$$

then (A15) and (A16) imply

$$V^{\phi^{a-1}}_{\max M_a}(b) > V^{\phi^{a-1}}_{\max M_a}(a). \tag{A17}$$

By Lemma 2, since $b < a$,

$$V^{\phi^{a-1}}(b) >_a V^{\phi^{a-1}}(a). \tag{A18}$$

We have, however,

$$V^{\phi^{a-1}}(a) = V^{\phi_2^{a-1}}(a) = u(a) + \beta V^{\phi_2^{a-1}}\left(\phi_2^{a-1}(a)\right) \geq_a u(a) + \beta V^{\phi_2^{a-1}}(a) = u(a) + \beta V^{\phi^{a-1}}(a)$$

($V^{\phi^{a-1}}(a) = V^{\phi_2^{a-1}}(a)$ by definition of ϕ^{a-1}, and the inequality holds because ϕ_2^{a-1} is MVE on $[a, m]$). Consequently, (A17) and (A18) imply (A14). (Notice that using $f(a) = 2$, we could similarly prove that there is $b \in [a+1, \phi^a(a+1)]$ such that (A14) holds.)

Let us now take state some quasi-median voter in state a, $j \in M_a$, and state $d \in B$ such that

$$d = \arg\max_{b \in B} V_j^*(b), \tag{A19}$$

and define monotone mapping $\phi : S \to S$ as

$$\phi(s) = \begin{cases} \phi_1^{a-1}(s) & \text{if } s < a \\ d & \text{if } s = a \\ \phi_2^a(s) & \text{if } s > a \end{cases}$$

(note that $V^\phi(s) = V^*(s)$ for $x \neq a$). Let us prove that ϕ is a MVE on S.

By construction of d (A19), we have that $b \in \left[\phi_1^{a-1}(a-1), \phi_2^a(a+1)\right] \cap F(a)$,

$$V^\phi(b) \not>_a V^\phi(d).$$

This is automatically true for $b \in B$, whereas if $b \notin F(a) \setminus B$ and $b \neq a$, the opposite would imply

$$V^\phi(b) >_a u(a) / (1-\beta),$$

which would contradict $b \notin B$; finally, if $b = a$,

$$V^\phi(a) >_a V^\phi(d)$$

is impossible, as this would imply

$$u(a) >_a (1-\beta) V^\phi(d)$$

contradicting (A14), given the definition of d (A19). Now, Lemma 5 implies that $\phi' = \phi|_{[1,a]}$ is a MVE on $[1, a]$.

Suppose, to obtain a contradiction, that ϕ is not MVE. Since ϕ is made from MVE ϕ' on $[1, a]$ and MVE ϕ_2^a on $[a+1, m]$, properties 1 and 3 of Definition 3 are satisfied, and by Lemma 4 there are only two possible monotone deviations that may prevent ϕ from being MVE. First, suppose that for some $y \in [a+1, \phi_2^a(a+1)] \cap F(a)$,

$$V^{\phi}(y) >_a V^{\phi}(d). \tag{A20}$$

However, this would contradict (A19) (and if $y \notin B$, then (A20) is impossible as $d \in B$). The second possibility is that for some $y \in [d, a]$,

$$V^{\phi}(y) >_{a+1} V^{\phi}(\phi_2^a(a+1)).$$

This means that

$$V^{\phi}_{\max M_{a+1}}(y) > V^{\phi}_{\max M_{a+1}}(\phi_2^a(a+1)).$$

At the same time, for any $x \in [a+1, \phi_2^a(a+1)] \cap F(a)$, we have

$$V^{\phi}_{\max M_{a+1}}(x) \le V^{\phi}_{\max M_{a+1}}(\phi_2^a(a+1))$$

(otherwise Lemma 2 would imply a profitable deviation to x). This implies that for any such x,

$$V^{\phi}_{\max M_{a+1}}(y) > V^{\phi}_{\max M_{a+1}}(x).$$

Now, recall that

$$\phi_1^a \in \arg \max_{\phi \in \Phi^{[1,a]}, z \in [\phi(a), a] \cap F(a)} V^{\phi}_{\max M_{a+1}}(z).$$

This means that there is $z \in [\phi_1^a(a), a] \cap F(a)$ such that

$$V^{\phi_1^a}_{\max M_{a+1}}(z) \ge V^{\phi}_{\max M_{a+1}}(y),$$

and thus for any $x \in [a+1, \phi_2^a(a+1)] \cap F(a)$,

$$V^{\phi_1^a}_{\max M_{a+1}}(z) > V^{\phi}_{\max M_{a+1}}(x).$$

But $\phi_1^a = \phi^a$ on the left-hand side, and $\phi = \phi^a$ on the right-hand side. We therefore have that the following maximum is achieved on $[\phi^a(a), a]$:

$$\arg \max_{z \in [\phi^a(a), \phi^a(a+1)] \cap F(a)} V^{\phi^a}_{\max M_{a+1}}(z) \subset [\phi^a(a), a],$$

i.e., that [3] in Lemma 4 holds. But this contradicts that $f(a) = 2$. This contradiction completes the induction step, which proves existence of MVE.

Finally, suppose that ϕ is a monotone MVE; take any s_0. If $\phi(s_0) \geq s_0$, then monotonicity implies $\phi^2(s_0) \geq \phi(s_0)$ etc, and thus the sequence $\left\{\phi^k(s_0)\right\}$ is weakly increasing in k. It must therefore have a limit. A similar reasoning applies if $\phi(s_0) < s_0$, which completes the proof. ■

Proof of Theorem 2. Part 1. Suppose that there are two MVE ϕ_1 and ϕ_2. Without loss of generality, assume that m is the minimal number of states for which this is possible, i.e., if $|S| < m$, then transition mapping is unique. Obviously, $m \geq 2$.

Consider the set $Z = \{x \in S \mid \phi_1(x) \neq \phi_2(x)\}$, and denote $a = \min Z$, $b = \max Z$. Without loss of generality, assume that ϕ_1 and ϕ_2 are enumerated such that $\phi_1(a) < \phi_2(a)$.

Let us first prove the following auxiliary result: $a < m$; $b > 1$; if $x \in [\max\{2, a\}, b]$, then $\phi_1(x) < x \leq \phi_2(x)$; if $x \in [a, \min\{b, m-1\}]$, then $\phi_1(x) \leq x < \phi_2(x)$.

To do this, we first show that if $\phi_1(x) = x$, then $x = 1$ or $x = m$. Indeed, assume the opposite and consider $\phi_2(x)$. If $\phi_2(x) < x$, then $\phi_1|_{[1,x]} \neq \phi_2|_{[1,x]}$ are two MVE for the set of states $[1, x]$, which contradicts the choice of m. If $\phi_2(x) > x$, we get a similar contradiction for $[x, m]$, and if $\phi_2(x) = x$, we get a contradiction by considering $[1, x]$ if $a < x$ and $[x, m]$ if $a > x$. Similarly, if $\phi_2(x) = x$, then either $x = 1$ or $x = m$.

Now assume, to obtain a contradiction, that $a = m$. Then $Z = \{m\}$, so $\phi_1|_{[1,m-1]} = \phi_2|_{[1,m-1]}$, and then having $\phi_1(m) \neq \phi_2(m)$ is impossible for generic parameter values. We would get a similar contradiction if $b = 1$, which proves that $a < m$ and $b > 1$, thus proving the first part of the auxiliary result.

Let us now show that for $x \in [a, b] \setminus \{1, m\}$, we have that either $\phi_1(x) < x < \phi_2(x)$ or $\phi_2(x) < x < \phi_1(x)$. Indeed, neither $\phi_1(x) = x$ nor $\phi_2(x) = x$ is possible. If $\phi_1(x) < x$ and $\phi_2(x) < x$, then $\phi_1|_{[1,x]}$ and $\phi_2|_{[1,x]}$ are two different MVE on $[1, x]$, which is impossible; we get a similar contradiction if $\phi_1(x) > x$ and $\phi_2(x) > x$. This also implies that if $a < x < b$, then $x \in Z$.

We now prove that for any $x \in Z$, $\phi_1(x) < \phi_2(x)$. Indeed, suppose that $\phi_2(x) > \phi_1(x)$ (equality is impossible as $x \in Z$); then $x > a \geq 1$. If $x < m$, then, as we proved, we must have $\phi_2(x) < x < \phi_1(x)$, and if $x = m$, then $\phi_2(x) < \phi_1(x) \leq m = x$. In either case, $\phi_2(x) < x$, and since $\phi_2(a) > \phi_1(a) \geq 1$, then by monotonicity of ϕ_2 there must be $y : 1 \leq a < y < x \leq m$ such that $\phi_2(y) = y$, but we proved that this is impossible. Hence, $\phi_1(x) < \phi_2(x)$ for any $x \in Z$, and using the earlier result, we have $\phi_1(x) < x < \phi_2(x)$ for any $x \in Z \setminus \{1, m\}$.

To finish the proof, it suffices to show that $\phi_1(1) = 1$ and $\phi_2(m) = m$. Suppose, to obtain a contradiction, that $\phi_1(1) > 1$. We then have $\phi_2(1) > 1$, then $\phi_1(2) \geq 2$ and $\phi_2(2) \geq 2$ and thus

$\phi_1|_{[2,m]}$ and $\phi_2|_{[2,m]}$ are MVE on $[2,m]$, and since $b \neq 1$, they must be different, which would again contradict the choice of m. We would get a similar contradiction if $\phi_2(m) = m$. This completes the proof of the auxiliary result.

To finish the proof of the Theorem, notice that the auxiliary result implies, in particular, that $Z = [a,b] \cap S$, so Z has no "gaps". We define function $g : Z \to \{1,2\}$ as follows. If $V_{M_x}^{\phi_1}(x) > V_{M_x}^{\phi_2}(x)$, then $g(x) = 1$, and if $V_{M_x}^{\phi_2}(x) > V_{M_x}^{\phi_1}(x)$, then $g(x) = 2$; the exact equality cannot hold generically. Intuitively g picks the equilibrium (left or right) that agent M_x prefers.

Let us prove that $g(a) = 2$ and $g(b) = 1$. Indeed, suppose that $g(a) = 1$; since $a < m$, we must have $\phi_1(a) \leq a < \phi_2(a)$ (with equality if $a = 1$ and strict inequality otherwise). Consider two cases. If $a > 1$, then for $x < a$, $\phi_1(x) = \phi_2(x)$, and since $\phi_1(a) < a$, then $V_{M_a}^{\phi_1}(\phi_1(a) \mid a) = V_{M_a}^{\phi_2}(\phi_1(a) \mid a)$. But $V_{M_x}^{\phi_1}(x) > V_{M_x}^{\phi_2}(x)$ implies that $V_{M_a}^{\phi_1}(\phi_1(a) \mid a) > V_{M_a}^{\phi_2}(\phi_2(a) \mid a)$ (provided that $\beta \neq 0$), and thus $V_{M_a}^{\phi_2}(\phi_1(a) \mid a) > V_{M_a}^{\phi_2}(\phi_2(a) \mid a)$, which contradicts that ϕ_2 is MVE. If $a = 1$, then $g(a) = 1$ would imply that $V_{M_1}^{\phi_1}(1) > V_{M_1}^{\phi_2}(1)$. But $\phi_1(1) = 1$, which means $\frac{u_{M_1}(1)}{1-\beta} > V_{M_1}^{\phi_2}(1)$, thus $u_{M_1}(1) + \beta V_{M_1}^{\phi_2}(1) > V_{M_1}^{\phi_2}(1)$. But $V_{M_1}^{\phi_2}(1) = u_{M_1}(1) + \beta V_{M_1}^{\phi_2}(\phi_2(1) \mid 1)$, and thus, provided that $\beta \neq 0$, we have $V_{M_1}^{\phi_2}(1 \mid 1) > V_{M_1}^{\phi_2}(\phi_2(1) \mid 1)$. This contradicts that ϕ_2 is an MVE, thus proving that $g(a) = 2$. We can similarly prove that $g(b) = 1$.

Clearly, there must be two states $s, s+1 \in Z$ such that $g(s) = 2$ and $g(s+1) = 1$. For such s, let us construct mapping ϕ as follows:

$$\phi(x) = \begin{cases} \phi_1(x) & \text{if } x \leq s \\ \phi_2(x) & \text{if } x > s \end{cases};$$

then $\phi(s) \leq s < \phi_2(s)$ (inequality is strict unless $s = 1$) and $\phi(s+1) \geq s+1 > \phi(s)$ (inequality is strict unless $s+1 = m$), which means that mapping ϕ is monotone. Now, $g(s) = 2$ implies that $u_{M_s}(x) + \beta V_{M_s}^{\phi_2}(\phi_2(s) \mid s) = V_{M_s}^{\phi_2}(s) > V_{M_s}^{\phi_1}(s) = u_{M_s}(x) + \beta V_{M_s}^{\phi_1}(\phi_1(s) \mid s)$. But $V_{M_s}^{\phi_2}(\phi_2(s) \mid s) = V_{M_s}^{\phi}(\phi_2(s) \mid s)$ and $V_{M_s}^{\phi_1}(\phi_1(s) \mid s) = V_{M_s}^{\phi}(\phi_1(s) \mid s)$, and thus $V_{M_s}^{\phi}(\phi_2(s) \mid s) > V_{M_s}^{\phi}(\phi_1(s) \mid s)$ (note also that $s+1 \leq \phi_2(s) \leq \phi_2(s+1)$). Similarly, $g(s+1) = 1$ implies $V_{M_{s+1}}^{\phi}(\phi_1(s+1) \mid s+1) > V_{M_{s+1}}^{\phi}(\phi_2(s+1) \mid s+1)$. But this contradicts Lemma 4 for mapping ϕ. This contradiction completes the proof.

Part 2. As in Part 1, we can assume that m is the minimal number of states for which this is possible. We can then establish, similarly to Part 1, that if $\phi_1(x) = x$, then $x = 1$ or $x = m$. If $\phi_1(x) < x < \phi_2(x)$ or vice versa, then for all $i \in M_x$, there must be both a state $x_1 < x$ and a state $x_2 > x$ such that $u_i(x_1) > u_i(x)$ and $u_i(x_2) > u_i(x)$, which contradicts the assumption in this case. Since for $1 < x < m$, $\phi(x) \neq x$, we get that $\phi_1(x) = \phi_2(x)$ for such x. Let us prove that $\phi_1(1) = \phi_2(1)$. If this is not the case, then $\phi_1(1) = 1$ and $\phi_2(1) = 2$ (or vice versa). If $m = 2$, then monotonicity implies $\phi_2(2) = 2$, and if $m > 2$, then, as proved earlier, we must have $\phi_2(x) = x + 1$ for $1 < x < m$ and $\phi_2(m) = m$. In both cases, we have $\phi_1(x) = \phi_2(x) > 1$ for $1 < x \leq m$. Hence, $V_i^{\phi_1}(2) = V_i^{\phi_2}(2)$ for all $i \in N$. Since ϕ_1 is MVE, we must have $u_i(1)/(1-\beta) \geq V_i^1(2)$ for $i \in M_1$, and since ϕ_2 is MVE, we must have $V_i^2(2) \geq u_i(1)/(1-\beta)$. Generically, this cannot hold, and this proves that $\phi_1(1) = \phi_2(1)$. We can likewise prove that $\phi_1(m) = \phi_2(m)$, which implies that $\phi_1 = \phi_2$. This contradicts the hypothesis of non-uniqueness. ■

Proof of Theorem 3. The existence is proved in the text. Since, on equilibrium path, there is only a finite number of shocks, then from some period t on, the environment will be the same, say E^x. Since ϕ_{E^x} is monotone, the sequence $\{s_t\}$ has a limit by Theorem 1. The fact that this limit may depend on the sequence of shocks realization is shown by Example 5. ■

Proof of Theorem 4. Part 1. Without loss of generality, suppose that h is the minimal number for which two monotone MVE $\phi = \{\phi_E\}_{E\in\mathcal{E}}$ and $\phi' = \{\phi'_E\}_{E\in\mathcal{E}}$ exist. If we take $\tilde{\mathcal{E}} = \{E_2, \ldots, E_h\}$ with the same environments $E_2, \ldots, E_h$ and the same transition probabilities, we will (generically) have a unique monotone MVE $\tilde{\phi} = \{\phi_E\}_{E\in\mathcal{E}'} = \{\phi'_E\}_{E\in\mathcal{E}'}$ by assumption. Now, with the help of transformation used in the proof of 3 we get that ϕ_{E_1} and ϕ'_{E_1} must be MVE in a certain stationary environment E'. However, by Theorem 2 such MVE is unique, which leads to a contradiction.

Part 2. The proof is similar to that of Part 1. The only step is that we need to verify that we can apply Part 2 of Theorem 2 to the stationary environment E'. In general, this will not be the case. However, it is easy to notice (by examining the proof of Part 2 of Theorem 2) that instead of single-peakedness, we could require a weaker condition: that for each $s \in S$ there is $i \in M_s$ such that there do not exist $x < s$ and $y > s$ such that $u_i(x) \geq u_i(s)$ and $u_i(y) \geq u_i(s)$.

We can now prove that if $\{u_i(s)\}_{i\in N}^{s\in S}$ satisfy this property and ϕ is MVE, then $\left\{V_i^\phi(s)\right\}_{i\in N}^{s\in S}$ also does. Indeed, suppose, to obtain a contradiction, that for some $s \in S$, for all $i \in M_s$ there are $x_i < s$ and $y_i > s$ such that $V_i^\phi(x_i) \geq V_i^\phi(s)$ and $V_i^\phi(y_i) \geq V_i^\phi(s)$; without loss of generality, we may assume that x_i and y_i minimize $|x_i - s|$ and $|y_i - s|$ among such x_i and y_i.

Consider the case $\phi(s) > s$. This implies that for all $i \in M_s$, there is $a > s$ such that $u_i(a) > u_i(s)$, and therefore for all $i \in M_s$ and all $a < s$, $u_i(z) < u_i(s)$. Moreover, for all $i \in M_s$, $u_i(z) < V_i^\phi(s)/(1-\beta)$. Take $j = \max M_s$, and let $z = x_j$. We cannot have $\phi(z) \leq z$, because then $V_j^\phi(\phi(z)) \geq V_j^\phi(s)$ would be impossible. Thus, $\phi(z) > z$, and in this case we must have $\phi(z) > s$, To see this, notice that $V_j^\phi(z) = u_j(z) + \beta V_j^\phi(\phi(z))$. If $\phi(z) < s$, then $V_j^\phi(z) \geq V_j^\phi(s)$ and $u_j(z) < V_i^\phi(s)/(1-\beta)$, implying $V_j^\phi(\phi(z)) > V_j^\phi(s)$ and thus contradicting the choice of $z = x_j$. If $\phi(z) = s$, then $V_j^\phi(z) = u_j(z) + \beta V_j^\phi(\phi(z))$ contradicts $V_j^\phi(z) \geq V_j^\phi(s)$ and $u_j(z) < V_i^\phi(s)/(1-\beta)$. Consequently, $\phi(z) > s$. Monotonicity of ϕ implies $s < \phi(z) \leq \phi(s)$. Now, $V_j^\phi(z) \geq V_j^\phi(s)$ and $u_j(z) < u_j(s)$ implies $V_j^\phi(\phi(z)) > V_j^\phi(\phi(s))$ (and in particular, $\phi(z) < \phi(s)$). Since $j = \max M_s$, we have $V^\phi(\phi(z)) >_s V^\phi(\phi(s))$. Since $s < \phi(z) < \phi(s)$, $\phi(z) \in F_s$, and therefore a deviation in s from $\phi(s)$ to $\phi(z)$ is feasible and profitable. This contradicts that ϕ is a MVE. We would get a similar contradiction if we assumed that $\phi(s) < s$.

Finally, assume $\phi(s) = s$. Then take any $i \in M_s$, and suppose, without loss of generality, that for any $a < s$, $u_i(a) < u_i(s)$. Then, since for all such a, $\phi^k(s) \leq s$ for all $k \geq 1$, we must have $V_i^\phi(a) < V_i^\phi(s)$, which contradicts the assertion. This proves the auxiliary result.

We have thus proved that under the assumptions of the Theorem, the environment constructed in the proof of 3 satisfies the requirements Part 2 of Theorem 2. The rest of the proof follows immediately. ■

Proof of Theorem 5. Part 1. It suffices to prove this result for stationary case. For each $s \in S$ take any protocol such that if $\phi(s) \neq s$, then $\theta_s(|F_s| - 1) = \phi(s)$ (i.e., the desired transition is the last one to be considered). We claim that there is a strategy profile σ such that if for state s, $\phi(s) = s$, then no alternative is accepted, and if $\phi(s) \neq s$, then no alternative is accepted until the last stage, and in this last stage, the alternative $\phi(s)$, is accepted.

Indeed, under such profile, the continuation strategies are given by (4). To show that such outcome is possible in equilibrium, consider first periods where $\phi(s) \neq s$. Consider the subgame reached if no alternatives were accepted before the last one. Since by property 3 of Definition 3, $V^\phi(\phi(s)) \geq_s V^\phi(s)$, it is a best response for players to accept $\phi(s)$. Let us now show, by backward induction, that if stage k, $1 \leq k \leq |F_s| - 1$ is reached without any alternatives accepted, then there is an equilibrium where $\phi(s)$ is accepted in the last stage. The base was just proved. The induction step follows from the following: if at stage k, alternative $y = \theta_s(k)$ is under consideration, then accepting it yields a vector of payoffs $V^\phi(y)$, and rejecting it yields, by induction, $V^\phi(\phi(s))$. Since by property 2 of Definition 3, $V^\phi(y) \not>_s V^\phi(\phi(s))$, it is a best response to reject the alternative y. Consequently, $\phi(s)$ will be accepted by induction. This proves the induction step, and therefore $\phi(s)$ is the outcome in a period which started with s. Now consider a period where $\phi(s) = s$. By backward induction, we can prove that there is an equilibrium where no proposal is accepted. Indeed, the last proposal $\theta_s(|F_s|-1)$ may be rejected, because $V^{\phi}(\theta_s(|F_s|-1)) \not>_s V^{\phi}(s)$ by property 2 of Definition 3. Going backward, if for some stage k, s is the outcome once $\theta_s(k)$ was rejected, sufficiently many players may reject $\theta_s(k)$, because $V^{\phi}(\theta_s(k)) \not>_s V^{\phi}(s)$. This proves that in periods where $\phi(s) = s$, it is possible to have an equilibrium where no proposal is accepted. Combining the equilibrium strategies for different initial s in the beginning of the period, we get a MPE which induces transition mappings $\phi(s)$.

Part 2. If the transition mapping is monotone, then continuation utilities $\left\{V^{\phi}_{E,i}(s)\right\}^{s\in S}_{i\in N} = \left\{V^{\sigma}_{E,i}(s)\right\}^{s\in S}_{i\in N}$ satisfy increasing differences for any $E \in \mathcal{E}$. Again, the proof that ϕ is MVE reduces to the stationary case. For each state s, we consider the set $J_s \subset \{1, \ldots, |F_s|-1\}$ of stages k where the alternative under consideration, $\theta_s(k)$, is accepted if this stage is reached. Naturally, $\phi(s) = s$ if and only if $J_s = \varnothing$, and if $J_s \neq \varnothing$, then $\phi(s) = \theta_s(\min J_s)$. Moreover, one can easily prove by induction that for any $j, k \in J_s$ such that $j \leq k$, $V^{\phi}(\theta_s(j)) \geq_s V^{\phi}(\theta_s(k))$ and for any $j \in J_s$, $V^{\phi}(\theta_s(j)) \geq_s V^{\phi}(s)$.

Take any $s \in S$. Property 1 of Definition 3 holds trivially, because only states in F_s are considered as alternatives and may be accepted. Let us show that property 2 holds. First, consider the case $\phi(s) = s$. Suppose, to obtain a contradiction, that for some $y \in F_s$, $V^{\phi}(y) >_s V^{\phi}(s)$. Suppose that this y is considered at stage k. But then, if stage k is reached, a winning coalition of players must accept y, because rejecting it leads to s. Then $k \in J_s$, contradicting $J_s = \varnothing$ for such s. Second, consider the case $\phi(s) \neq s$. Again, suppose that for some $y \in F_s$, $V^{\phi}(y) >_s V^{\phi}(\phi(s))$; notice that $y \neq s$, because $V^{\phi}(\phi(s)) = V^{\phi}(\theta_s(\min J_s)) \geq_s V^{\phi}(s)$. Let k be the stage where y is considered. If $k < \min J_s$, so y is considered before $\phi(s)$, then a winning coalition must accept y, which implies $k \in J_s$, contradicting $k < \min J_s$. If, on the other hand, $k > \min J_s$, then notice that $k \notin J_s$ (otherwise, $V^{\phi}(y) >_s V^{\phi}(\phi(s))$ is impossible). If $k > \max J_s$, then we have $V^{\phi}(y) >_s V^{\phi}(\phi(s)) = V^{\phi}(\theta_s(\min J_s)) \geq_s V^{\phi}(s)$, which means that this proposal must be accepted, so $k \in J_s$, a contradiction. If $k < \max J_s$, then we can take $l = \min\{J_s \cap [k+1, |F_s|-1]\}$. Since $V^{\phi}(y) >_s V^{\phi}(\phi(s)) = V^{\phi}(\theta_s(\min J_s)) \geq_s V^{\phi}(\theta_s(l))$, it must again be that y is accepted, so $k \in J_s$, again a contradiction. In all cases, the assertion that such y exists leads to a contradiction, which completes the proof.

Finally, we need to show that Property 3 of Definition 3 holds. This is trivial if $\phi(s) = s$. Otherwise, we already proved that for all $j \in J_s$, $V^{\phi}(\theta_s(j)) \geq_s V^{\phi}(s)$. In particular, this is true for $j = \min J_s$. Consequently, $V^{\phi}(\phi(s)) \geq_s V^{\phi}(s)$. This completes the proof that ϕ is a

MVE. ■

Proof of Theorem 6. Suppose, to obtain a contradiction, that $\phi_2(x) < x$. Then $\phi_1|_{S'}$ and $\phi_2|_{S'}$ are mappings from S' to S' such that both are MVE. Moreover, they are different, as $\phi_1(x) = x > \phi_2(x)$. However, this would violate the assumed uniqueness (either assumption needed for Theorem 2 continues to hold if the domain is restricted), which completes the proof. ■

Proof of Corollary 1. Consider an alternative set of environments $\mathcal{E}' = \{E_0, E_2\}$, where E_0 coincides with E_2 on S, but the transition probabilities are the same as in $\mathcal{E}$. Clearly, ϕ' such that $\phi'_{E_0} = \phi'_{E_2} = \phi_{E_2}$ is a MVE in $\mathcal{E}'$. Let us now consider stationary environments $\tilde{E}_0$ and $\tilde{E}_1$ obtained from $\mathcal{E}'$ and $\mathcal{E}$, respectively, using the procedure from the proof of Theorem 3. Suppose, to obtain a contradiction, that $\phi_{E_2}(x) < x$, then environments $\tilde{E}_0$ and $\tilde{E}_1$ coincide on $[1, s]$ by construction. Theorem 6 then implies that, since $\phi_{E_1}(x) = x$, then $\phi'_{E_0}(x) \geq x$ (since ϕ'_{E_0} and ϕ_{E_1} are the unique MVE in $\tilde{E}_0$ and $\tilde{E}_1$, respectively). But by definition of ϕ', $x < \phi'_{E_0}(x) = \phi_{E_2}(x) \leq x$, a contradiction. This contradiction completes the proof. ■

Proof of Theorem 7. Let us first prove this result for the case where each QMV is a singleton. Both before and after the shock, the mapping that would map any state x to a state which maximizes the stage payoff $u_{M_x}(y)$ would be a monotone MVE for $\beta < \beta_0$. By uniqueness, ϕ_{E_1} and ϕ_{E_2} would be these mappings under E_1 and E_2, respectively. Now it is clear that if the shock arrives at period t, and the state at the time of shock is $x = s_{t-1}$, then $\phi_{E_2}(x)$ must be either the same as $\phi_{E_1}(x)$ or must satisfy $\phi_{E_2}(x) > s$. In either case, we get a monotone sequence after the shock. Moreover, the sequence is the same if $s_\tau \leq s$, and if $s_\tau > s$, then we have $s_\tau > s \geq \tilde{s}_\tau$ automatically.

The general case may be proved by observing that a mapping that maps each state x to an alternative which maximizes by $u_{\min M_x}(y)$ among the states such that $u_i(y) \geq u_i(x)$ for all $i \in M_x$ is a monotone MVE. Such mapping is generically unique, and by the assumption of uniqueness it coincides with the mapping ϕ_{E_1} if the environment is E_1 and it coincides with ϕ_{E_2} if the environment is E_2. The remainder of the proof is analogous. ■

Proof of Theorem 8. It is sufficient, by transitivity, to prove this Theorem for the case where $\max M_{E_1,x} \neq \max M_{E_2,x}$ for only one state $x \in [s+1, m]$. Moreover, without loss of generality, we can assume that $\max M_{E_1,x} < \max M_{E_2,x}$. Notice that if $\phi_1(x) \geq x$, then ϕ_1 is MVE in environment E_2, and by uniqueness must coincide with ϕ_2.

Consider the remaining case $\phi_1(x) < x$; it implies $\phi_1(x-1) \leq x-1$. Consequently, $\phi_1|_{[1,x-1]}$ is MVE under either environment restricted on $[1, x-1]$ (they coincide on this interval). Suppose, to obtain a contradiction, that $\phi_1|_{[1,s]} \neq \phi_2|_{[1,s]}$; since $x > s$, we have $\phi_1|_{[1,x-1]} \neq \phi_2|_{[1,x-1]}$. We must then have $\phi_2(x-1) > x-1$ (otherwise there would be two MVE $\phi_1|_{[1,x-1]}$ and $\phi_2|_{[1,x-1]}$ on $[1, x-1]$, and therefore $\phi_2(x) \geq x$. Consequently, $\phi_2|_{[x,m]}$ is MVE on $[x, m]$ under environment E_2 restricted on $[x, m]$. Let us prove that $\phi_2|_{[x,m]}$ is MVE on $[x, m]$ under environment E_1 restricted on $[x, m]$ as well. Indeed, if it were not the case, then there must be a monotone deviation, as fewer QMV (in state x) imply that only property 2 of Definition 3 may be violated. Since under E_1, state x has fewer quasi-median voters than under E_2, it is only possible if $\phi_2(x) > x$, in which case $\phi_2(x+1) \geq x+1$. Then $\phi_2|_{[x+1,m]}$ would be MVE on $[x+1, m]$, and by Lemma 5 we could get MVE $\tilde{\phi}_2$ on $[x, m]$ under environment E_1. This MVE $\tilde{\phi}_2$ would be MVE on $[x, m]$ under environment E_2. But then under environment E_2 we have two MVE, $\tilde{\phi}_2$ and $\phi_2|_{[x,m]}$ on $[x, m]$, which is impossible.

We have thus shown that $\phi_1|_{[1,x-1]}$ is MVE on $[1, x-1]$ under both E_1 and E_2, and the same is true for $\phi_2|_{[x,m]}$ on $[x, m]$. Take mapping ϕ given by

$$\phi(y) = \begin{cases} \phi_1(y) & \text{if } y < x \\ \phi_2(y) & \text{if } y > x \end{cases}.$$

Since $\phi_1|_{[1,x-1]} \neq \phi_2|_{[1,x-1]}$ and $\phi_1|_{[x,m]} \neq \phi_2|_{[x,m]}$ ($\phi_1(x-1) \leq x-1$, $\phi_2(x-1) > x-1$, $\phi_1(x) < x$, $\phi_2(x) \geq x$), ϕ is not MVE in E_1 nor it is in E_2. By Lemma 4, in both E_1 and E_2 only one type of monotone deviation (at $x-1$ to some $z \in [x, \phi_2(x)]$ or at x to some $z \in [\phi_1(x-1), x]$) is possible. But the payoffs under the first deviation are the same under both E_1 and E_2; hence, in both environments it is the same type of deviation.

Suppose that it is the former deviation, at $x-1$ to some $z \in [x, \phi_2(x)]$. Consider the following restriction on feasible transitions:

$$\tilde{F}(a) = \begin{cases} F(a) & \text{if } a \geq x; \\ F(a) \cap [1, x-1] & \text{if } a < x; \end{cases}$$

denote the resulting environments by E_1 and E_2. This makes the deviation impossible, and thus ϕ is MVE in E_1 (in E_2 as well). However, ϕ_1 is also MVE in E_1, as it is not affected by the change is feasibility of transitions, and this contradicts uniqueness. Finally, suppose that the deviation is at x to some $z \in [\phi_1(x-1), x]$. Then consider the following restriction on feasible transitions:

$$\bar{F}(a) = \begin{cases} F(a) & \text{if } a < x; \\ F(a) \cap [x, m] & \text{if } a \geq x; \end{cases}$$

denote the resulting environments by $\bar{E}^1$ and $\bar{E}^2$. This makes the deviation impossible, and thus ϕ is MVE in E_2. However, ϕ_2 is also MVE in E_1, as it is not affected by the change in feasibility. Again, this contradicts uniqueness, which completes the proof. ∎

Proof of Proposition 1. Part 1. We start by proving that there exists a unique monotone MVE. To show this, we need to establish that all requirements for existence and generic uniqueness are satisfied.

(Increasing differences) Consider player i and take two states x, y with $x > y$. The policy in state x is b_{M_x} and in state y, it is b_{M_y}. Since $M_x \geq M_y$ and b is increasing in the identity of the player, we have $b_{M_x} \geq b_{M_y}$. Take the difference

$$\begin{aligned} u_i(x) - u_i(y) &= -(b_{M_x} - b_i)^2 - \sum\nolimits_{j \notin H_x} \gamma_j C_j - \left(-\left(b_{M_y} - b_i\right)^2 - \sum\nolimits_{j \notin H_y} \gamma_j C_j \right) \\ &= \left(b_{M_x} - b_{M_y}\right)\left(2b_i - b_{M_x} - b_{M_y}\right) - \sum\nolimits_{j \notin H_x} \gamma_j C_j + \sum\nolimits_{j \notin H_y} \gamma_j C_j. \end{aligned}$$

This only depends on i through b_i, which is increasing in b_i. Hence, increasing differences is satisfied.

(Monotone QMV) The QMV in state s is M_s. If $s \geq 0$, then an increase in s implies that players on the right get more power, and $s \leq 0$, then a decrease in s implies that players on the left get more power.

(Feasibility) All transitions are feasible, and thus the assumption holds trivially.

(QMV are singletons) This holds generically, when no two disjoint sets of players have the same power.

This establishes that there is a unique monotone MVE. To show that $\phi(0) = 0$, suppose not. Without loss of generality, $\phi(0) > 0$. Then if $s_1 = 0$, monotonicity implies that $s_t > 0$ for all $t > 1$. But $M_0 = 0$, thus $b_{M_0} = b_0$ and $u_{M_0}(0) = 0$, while $u_{M_0}(s) < 0$ for $s \neq 0$. This shows that if $\phi(0) > 0$, there is a profitable deviation to 0. This contradiction completes the proof.

Part 2. Consider the case $s < 0$ (the case $s > 0$ is considered similarly). Since $\phi(0) = 0$, monotonicity implies that $\phi(s) \leq 0$. To show that $\phi(s) \geq s$, suppose, to obtain a contradiction, that $\phi(s) < s$. Then, starting from the initial state $s_1 = s$, the equilibrium path will involve $s_t < s$ for all $t > 1$. Notice, however, that for the QMV M_s, $u_{M_s}(s) = -\sum_{j \notin H_s} \gamma_j C_j$, and for $x < s$, $u_{M_s}(x) = -(b_{M_x} - b_{M_s}) - \sum_{j \notin H_x} \gamma_j C_j < u_{M_s}(s)$, as H_x is a strict superset of H_s. Again, there is a profitable deviation, which completes the proof.

Part 3. Consider the mapping ϕ such that $\phi(s) = 0$ for all s. Under this mapping, continuation utilities are given by

$$V_i^{\phi}(s) = -(b_{M_s} - b_i)^2 - k \sum\nolimits_{j \notin H_s} \gamma_j C_j^* - \frac{\beta}{1-\beta}(b_0 - b_i)^2 .$$

Now, the two conditions required to hold for ϕ to be an MVE simplify to:

$$\begin{aligned} \text{for any } s, x &: \quad V_{M_s}^{\phi}(0) \geq V_{M_s}^{\phi}(x); \\ \text{for any } s &: \quad V_{M_s}^{\phi}(0) \geq V_{M_s}^{\phi}(s); \end{aligned}$$

clearly, the second line of inequalities is a subset of the first. This simplifies to

$$\text{for any } s, x\text{: } k \sum\nolimits_{j \notin H_x} \gamma_j C_j^* \geq (b_{M_s})^2 - (b_{M_x} - b_{M_s})^2 .$$

Clearly, as k increases, the number of equations that are true weakly increases. Furthermore, for k high enough, the left-hand side becomes arbitrarily large for all x except for $x = 0$ where it remains zero, but for $x = 0$, $b_{M_x} = 0$ and thus the right-hand side is zero as well. Finally, if k is small enough, the left-hand side is arbitrarily close to 0 for all s and x, and thus the inequality will be violated, e.g., for $s = x = 1$. This proves that there is a unique positive k^* with the required property. ■

Proof of Proposition 2. **Part 1.** The equilibrium exists and is unique because the required properties hold in each of the environments, and thus Theorems 3 and 4 are applicable.

Let ϕ_{E_f} be the mapping after radicals have left. Since the environment E_f allows for no further stochastic shocks, ϕ_{E_f} coincides with ϕ from Proposition 1 (i.e., if radicals are impossible). Now take any radical environment R_z (so states $x \leq z$ are controlled by radicals). Notice that $\phi_{R_z}(s)$ is the same for all $s \leq z$ (otherwise, setting $\phi_{R_z}(s) = \phi_{R_z}(z)$ for all $s < z$ would yield another MVE, thus violating uniqueness). Consider two situations: $z < 0$ and $z \geq 0$.

Suppose first that $z < 0$. Then $\phi_{R_z}(0) = 0$ (similar to the proof of Part 1 of Proposition 1), and thus by monotonicity $\phi_{R_z}(s) \in [-l-r, 0]$. For any x such that $z < x < 0$, $\phi_{R_z}(x) \geq x$ (again, similar to that proof). Notice that as b_{-l} varies, the mapping $\phi_{R_z}|_{[z+1, l+r]}$ does not change. Indeed, equilibrium paths starting from $x \geq z+1$ remain within that range, and thus continuation utilities of M_x for any $x \geq z+1$ do not depend on b_{-l}; moreover, a deviation from

$x \geq z+1$ to some $y \leq z$ cannot be profitable for obvious reasons. The state $\phi_{R_z}(z)$ is such that it maximizes the continuation utility of the radical $-l$ among the following alternaties: moving to some state $y \leq z$, staying there until transition to environment E_f and moving according to ϕ_{E_f}, and moving to some state $y > z$, moving according to ϕ_{R_z} until the transition to E_f and according to ϕ_{E_f} after the transition. Notice that as b_{-l} decreases, the continuation utilities of the radical $-l$ under all these options, except of moving to state $y = -l-r$, strictly decrease, while the payoff of that option remains unchanged (and equal to $-\frac{1}{1-\beta}k\sum_{j>-l}\gamma_j C_j^*$). Hence, a decrease in b_{-l} makes this transition more likely starting from state z, and thus for all $s \leq z$.

Now suppose that $z \geq 0$. Trivially, we must have $\phi_{R_z}(z) \leq 0$. In this case, $\phi_{R_z}|_{[z+1,l+r]}$ may depend on b_{-l}, moving to $y \in [z+1, l+r]$ is suboptimal for the radical anyway. So in this case, the equilibrium $\phi_{R_z}(z)$ maximizes the radical's continuation utility among the options of moving to some $y \leq 0$, staying there until transition to E_f, and then moving according to ϕ_{E_f}. Again, only for $y = -l-r$ the continuation payoff remains unchanged as b_{-l} decreases, and for all other options it decreases. Hence, in this case, too, a lower b_{-l} makes $\phi_{R_z}(z) = -l-r$ more likely. Moreover, since the equilibrium path starting from any $y \leq 0$ will only feature states $s \leq 0$, and for all possible $y \leq 0$, the path for lower y is first-order stochastically dominated by the path for higher y, an increase in k makes $\phi_{R_z}(z) = -l-r$ less likely.

It remains to prove that an increase in z decreases the chance of transition to $-l-r$ for any given $s \leq z$. This is equivalent to saying that a higher z decreases the chance that $\phi_{R_z}(-l-r) = -l-r$. Suppose that z increases by one. If $z \geq 0$ (thus increasing to $z+1 \geq 1$), then $\phi_{R_z}(-l-r)$ does not change as moving to $y \geq 1$ was dominated anyway. If $z < 0$ (thus increasing to $z+1 \leq 0$), then this increase does not change $\phi_{R_z}|_{[z+2,l+r]}$, and thus the only change is the option to stay in $z+1$ as long as the shock leading to E_f does not arrive. This makes staying in $-l-r$ weakly less attractive for the radical, and for some parameter values may make him switch.

Part 2. Suppose, to obtain a contradiction, that for some $s \leq 0$, $\phi_{E_1}(s) < s$. Without loss of generality we may assume that this is the lowest such s, meaning $\phi_{E_1}(s)$ is ϕ_{E_1}-stable. Consider a deviation at s from $\phi_{E_1}(s)$ to s. This deviation has the following effect on continuation utility. First, in the period of deviation, the QMV M_s gets a higher state payoff. Second, the continuation utilities if a transition to R_z for some z takes place immediately after that may differ (if there is no shock, then both paths will converge at $\phi_{E_1}(s)$ thus yielding the same continuation utilities). Now consider two cases: if $z \geq s$, then the radicals are in power in both

s and $\phi_{E_1}(s)$. As showed in the proof of Part 1, the radicals will transit to the same state, thus resulting in the same path and continuation utilities. If, however, $z < s$, then the transition in R_z will be chosen by M_s if he stayed in s, hence, this transition will maximize his continuation payoff under R_z, and this need not be true if he moved to $\phi_{E_1}(s)$ (regardless of whether or not radicals rule in this state). In all cases, the continuation utility after the current period is weakly higher if he stayed in s than if he moved to $\phi_{E_1}(s) < s$, and taking into account the first effect, we have a strictly profitable deviation. This contradicts the definition of MVE, which completes the proof. ■

Proof of Proposition 3. Part 1. Suppose, to obtain a contradiction, that $\phi_{E_1}(s) \leq x$ for all $x \geq 0$. By Part 2 of Proposition 2, $\phi_{E_1}(s) \geq s$ for $s \leq 0$, which now implies $\phi_{E_1}(0) = 0$.

Part 2. As in Theorem 3, we may treat the environment E_1 as static, with $W_i(s)$ as quasi-utilities and $\tilde{\beta} = \beta(1-\mu)$ as the discount factor. Assume, to obtain a contradiction, that for all $x \geq 0$, $\phi_{E_1}(s) \leq s$. The payoff from staying in 0 for player $M_0 = 0$ is $V_0(0) = \frac{W_0(0)}{1-\tilde{\beta}}$. By definition of MVE, $V_{M_s}\left(\phi_{E_1}(s)\right) \geq V_{M_s}(s)$, and since continuation utilities satisfy increasing differences, $\phi_{E_1}(s) \leq s$, and $M_0 \leq M_s$, it must be that $V_0\left(\phi_{E_1}(s)\right) \geq V_0(s)$. Since $V_0(s) = W_0(s) + \tilde{\beta}V_0\left(\phi_{E_1}(s)\right)$, we have $V_0\left(\phi_{E_1}(s)\right) \geq \frac{W_0(s)}{(1-\tilde{\beta})}$. Consequently, it must be that $V_0\left(\phi_{E_1}(s)\right) > V_0(0)$. This is impossible if $\phi_{E_1}(s) = 0$, and it suggests a profitable deviation at 0 from 0 to s otherwise. This contradiction proves that such x exists.

Part 3. Suppose, to obtain a contradiction, that for some $s > 0$, $\phi_{E_1}(s) > s$. Without loss of generality, assume that $\phi_{E_1}(s)$ is itself ϕ_{E_1}-stable. By definition of MVE, $V_{M_s}\left(\phi_{E_1}(s)\right) \geq V_{M_s}(s)$. This is equivalent to $\frac{W_{M_s}\left(\phi_{E_1}(s)\right)}{1-\tilde{\beta}} \geq W_{M_s}(s) + \frac{\tilde{\beta}W_{M_s}\left(\phi_{E_1}(s)\right)}{1-\tilde{\beta}}$, thus implying $W_{M_s}\left(\phi_{E_1}(s)\right) \geq W_{M_s}(s)$. Setting $y = \phi_{E_1}(s)$ and $x = s$, we have $y > x \geq 0$ and $W_{M_x}(y) \geq W_{M_x}(x)$, a contradiction. This completes the proof. ■

Proof of Proposition 4. Proposition 1 proved this result for environment E_f. For any of the radical environments R_z $(z < 0)$, the quasi-utility of the QMV of state 0, player 0, is $\tilde{u}_{R_z,0}(0) = 0$, and for $s \neq 0$, $\tilde{u}_{R_z,0}(s) < 0$. This means that continuation utility $\tilde{V}_{R_z,0}(s) < 0$. Hence, if $\phi_{R_z}(0) = s \neq 0$, there would be a profitable deviation at 0 from s to 0; this proves that $\phi_{R_z}(0) = 0$. Now, monotonicity yields that $\phi_{R_z}(s) \geq s$ for all $s \geq 0$. This tells us that if we consider $R_z|_{[0,l+r]}$ to be a static environment with quasi-utilities $\tilde{u}_{R_z,i}(s)$ and the quasi-discount factor $\tilde{\beta} = \beta(1-\nu)$, then $\phi_{R_z}|_{[0,l+r]}$ is an MVE. But notice that $\phi_{E_f}|_{[0,l+r]}$ is also an MVE in this environment, because continuation utilities $\tilde{V}_{R_z,i}(s)$ would equal the corresponding continuation utilities in the environment E_f, where it is an MVE: $\tilde{V}_{R_z,i}(s) = V_{E_f,i}(s)$. Since the MVE must be unique, we have $\phi_{R_z}|_{[0,l+r]} = \phi_{E_f}|_{[0,l+r]}$, and thus $\phi_{R_z}(s) \in [0,s]$ for $s \geq 0$, because this property holds for ϕ_{E_f}. Another iteration of this argument would establish the same for the initial environment E_1, which completes the proof. ■

Proof of Proposition 5. This is an immediate corollary of Theorem 6. ■

Proof of Proposition 6. Part 1. Suppose not; so there are $s \geq 0$ such that $\phi_{E_1}(s) = 0$ and $\phi_{E_f}(s) \neq 0$. Since $\phi_{E_f}(0) = 0$, we must have $s > 0$ and $\phi_{E_f}(s) > 0$. Without loss of generality, assume that s is the minimal of such $s > 0$. If $\phi_{E_f}(s) > 0$, it must be that there is some $x > 0$ such that $u_{M_s}(x) \geq u_{M_s}(0)$, and generically, this means that $u_{M_s}(x) > u_{M_s}(0)$. Moreover, there is such x that satisfies $0 < x \leq s$ (because for $x > 0$, $u_{M_s}(x) < u_{M_s}(s)$). But then $\phi_{E_1}(x) = 0$; this implies that in environment E_1 and state s, a deviation from 0 to x is profitable for group M_s. This contradiction completes the proof.

Part 2. Let us first prove that for any R_z and any $x \geq 0$, $\phi_{R_z}(x) \leq x$. Suppose, to obtain a contradiction, that $\phi_{R_z}(x) > x \geq 0$. Consider two cases. If $z \geq x$ (so radicals are in power), then at x they have a profitable deviation from $\phi_{R_z}(x)$ to x, since the path starting at x is first-order stochastically dominated by one starting at $\phi_{R_z}(x) > x$, both are contained in $[0, l+r]$, and on this set the preferences are radicals are monotone. Consequently, in this case, $\phi_{R_z}(x) > x$ is impossible. The second case is $z < x$, meaning that M_x is the QMV. In that case deviation to x is again profitable: indeed, $V_{E_f,M_x}(x)$ is maximal among all $V_{E_f,M_x}(y)$ for $y \geq x$, and the path $\phi_{R_z}(x), \phi^2_{R_z}(x), \ldots$ yields, pointwisely, lower utility than the path $x, \phi_{R_z}(x), \phi^2_{R_z}(x), \ldots$. This shows that $\phi_{R_z}(x) \leq x$.

Now suppose that $x \geq 0$ is stable in E. Then it does not change if a shock never arrives, and the result holds trivially. Once a transition to R_z has taken place, we have $\phi_{R_z}(x) \leq x$, implying that the entire path satisfies this property. If there is never a transition to E_f, then the statement again holds; otherwise, suppose that this shock arrives when the society is at $s \leq x$. Since $\phi_{E_f}(x) \leq x$, we must have that $\phi_{E_f}(s) \leq x$, and so the entire path lies below x. Convergence follows from finiteness of S, and the ultimate state y satisfies $y \leq x$. ■

Proof of Proposition 7. This result is a direct corollary of Theorem 8. ■

Proof of Proposition 8. All the Assumptions hold for trivial reasons, however, we need to verify that the increasing differences (Assumption 2) hold when one of the agents is group

$-l$. Take another group $x > -l$; we have

$$u_s(x) - u_s(-l) = \begin{cases} (b_x - b_{-l})\left(2b_{M_s} - b_{M_x} - b_{M_{-l}}\right) - (1-\rho)\sum_{j \notin H_s} \gamma_j C_j & \text{if } s < 0 \\ (b_x - b_{-l})\left(2b_{M_s} - b_{M_x} - b_{M_{-l}}\right) & \text{if } s \geq 0 \end{cases}.$$

But $b_{M_s}$is increasing in s, and $\sum_{j \notin H_s} \gamma_j C_j$ is decreasing while remaining positive. This implies that $u_s(x) - u_s(-l)$ is increasing in s, so all Assumptions hold.

Take some ρ and ρ' such that $\rho > \rho'$. Suppose, to obtain a contradiction, that $\phi_{E_1}(0) > 0$, but $\phi'_{E_1}(0) = 0$. Since radicals cannot come to power at state 1, we must have $\phi_{E_1}(1) \in \{0, 1\}$, and $\phi'_{E_1}(1) \in \{0, 1\}$. We therefore have $\phi_{E_1}(0) = \phi_{E_1}(1) = 1$.

It is easy to check that for any radical environment R_z and for any $x \leq z$, $\phi'_{R_z}(x) \leq \phi_{R_z}(x) \leq 0$, and therefore, if in period t, the environment is R_z and the state is $s_t = s'_t \leq z$, then for all $\tau \geq t$ and for all realizations of shocks, we have $s'_\tau \leq s_\tau \leq 0$. From this, we have that $V_{R_z,0}(0) = V'_{R_z,0}(0)$ and $V_{R_z,0}(1) = V'_{R_z,0}(1)$ whenever $z < 0$ (indeed, the equilibrium paths in these cases in R_z and E_f are the same and do not involve states $x < 0$).

Notice also that the mapping $\phi_{R_z}|_{[0,r]} = \phi_{E_f}|_{[0,r]}$ for $z < 0$. Denote $\lambda^* = \mu_{-l-r} - \mu_0$, so λ^* is the probability of a shock to a radical environments other than R_0.

Let us prove that $\phi_{E_1}(0) = 1$ implies $\phi_{R_0}(0) = 1$. Indeed, from $\phi_{E_1}(0) = 1$, we have $\tilde{u}_{E_1,0}(1) \geq \tilde{u}_{E_1,0}(0)$. By definition,

$$\begin{aligned} \tilde{u}_{E_1,0}(1) &= u_0(1) + \beta\left(\lambda^* V_{E_f,0}(1) + \lambda_0 V_{R_0,0}(1)\right), \\ \tilde{u}_{E_1,0}(0) &= u_0(0) + \beta\left(\lambda^* V_{E_f,0}(0) + \lambda_0 V_{R_0,0}(0)\right). \end{aligned}$$

But $u_0(1) < u_0(0)$ and, clearly, $V_{E_f,0}(1) < 0 = V_{E_f,0}(0)$. This means $V_{R_0,0}(1) > V_{R_0,0}(0)$, implying that $\phi_{R_0}(0) = 1$ (which in turn implies $\phi_{R_0}(1) = 1$).

Now, notice that we have similar formulas for $\tilde{u}_{E_1,0}(1)$ and $\tilde{u}_{E_1,0}(0)$, and moreover, $V_{E_f,0}(1) = V'_{E_f,0}(1)$ and $V_{E_f,0}(0) = V'_{E_f,0}(0)$. Therefore,

$$\begin{aligned} \tilde{u}_{E_1,0}(1) - \tilde{u}'_{E_1,0}(1) &= \beta\lambda_0\left(V_{R_0,0}(1) - V'_{R_0,0}(1)\right), \\ \tilde{u}_{E_1,0}(0) - \tilde{u}'_{E_1,0}(0) &= \beta\lambda_0\left(V_{R_0,0}(0) - V'_{R_0,0}(0)\right). \end{aligned}$$

But $\phi_{R_0}(0) = \phi_{R_0}(1) = 1$ implies $V_{R_0,0}(1) = V'_{R_0,0}(1)$. On the other hand, $V_{R_0,0}(0) \geq V'_{R_0,0}(0)$. Together, this all implies that

$$\left(\tilde{u}_{E_1,0}(1) - \tilde{u}'_{E_1,0}(1)\right) - \left(\tilde{u}_{E_1,0}(0) - \tilde{u}'_{E_1,0}(0)\right) \leq 0.$$

Since $\tilde{u}_{E_1,0}(1) \geq \tilde{u}_{E_1,0}(0)$, it must be that $\tilde{u}'_{E_1,0}(1) \geq \tilde{u}'_{E_1,0}(0)$. This means $\tilde{u}'_{E_1,M_1}(1) \geq \tilde{u}'_{E_1,M_1}(0)$, implying $\phi'_{E_1}(1) = 1$. But then $\tilde{u}'_{E_1,0}(1) \geq \tilde{u}'_{E_1,0}(0)$ is incompatible with $\phi'_{E_1}(0) = 0$. This contradicts our initial assertion, which completes the proof. ■

Proof of Theorem 9. Part 1. It suffices to prove this result in stationary environments. By Theorem 10, there are no cycles, and thus for any $x \in S$, the sequence $x, \phi(x), \phi^2(x), \ldots$ has a limit. Suppose, to obtain a contradiction, that MVE ϕ is nonmonotone, which means there are states $x, y \in S$ such that $x < y$ and $\phi(x) > \phi(y)$. Without loss of generality we can assume that x and y are such that the set $Z = \left\{x, \phi(x), \phi^2(x), \ldots; y, \phi(y), \phi^2(y), \ldots\right\}$ has fewest different states. In that case, mapping ϕ is monotone on the set $Z \setminus \{x, y\}$, which implies that $\{V_i^s\}_{i\in N}^{s\in Z\setminus\{x,y\}}$ satisfies increasing differences. By property 2 of Definition 3 applied to state x, we get

$$V_{\max M_x}(\phi(x)) \geq V_{\max M_x}(\phi(y)), \tag{A21}$$

and if we apply it to state y,

$$V_{\min M_y}(\phi(y)) \geq V_{\min M_y}(\phi(x)). \tag{A22}$$

Since $\max M_x \leq \min M_y$ by assumption, (A21) implies

$$V_{\min M_y}(\phi(x)) \geq V_{\min M_y}(\phi(y)).$$

Since in the generic case inequalities are strict, this contradicts (A22).

Part 2. Again, consider stationary environments only. If ϕ is nonmonotone, then for some $x, y \in S$ we have $x < y$ and $\phi(x) > \phi(y)$, which in this case implies $\phi(x) = y = x + 1$ and $\phi(y) = x$. But by Theorem 10, this is generically impossible. This contradiction completes the proof. ■

Proof of Theorem 10. Let us first rule out cycles, where for some x, $\phi(x) \neq x$, but $\phi^k(x) = x$ for some $k > 1$. Without loss of generality, let k be the minimal one for which this is true, and x be the highest element in the cycle. In this case, the we have, for any $i \in N$,

$$\begin{aligned} V_i(x) - V_i(\phi(x)) &= u_i(x) + \beta V_i(\phi(x)) - V_i(\phi(x)) = u_i(x) - (1-\beta) V_i(\phi(x)) \\ &= \sum\nolimits_{j=1}^{k-1} \frac{(1-\beta)\beta^{j-1}}{1-\beta^k} \left(u_i(x) - u_i\left(\phi^j(x)\right)\right), \end{aligned}$$

which is increasing in i, since each term is increasing in i as $x > \phi^j(x)$ for $j = 1, \ldots, k-1$. This means that $\{V_i(s)\}_{i\in N}^{s\in\{\phi(x),x\}}$ satisfies the increasing differences. Because of that, property 3 of Definition 3, when applied to state x, implies that $V_i(\phi(x)) \geq V_i(x)$ for all $i \in M_x$. However, if we take $y = \phi^{k-1}(x)$ (so $\phi(y) = x$), then property 2 of Definition 3 would imply that $V_i(x) \geq V_i(\phi(x))$ for at least one $i \in M_y$. Increasing differences implies that $V_i(x) \geq V_i(\phi(x))$

for at least one $i \in M_x$, and therefore for such i, $V_i(x) = V_i(\phi(x))$. Generically, this is impossible, which implies that cycles are generically ruled out.

Now, to prove that any path is monotone, assume the opposite, and take x that generates the shortest nonmonotone path (i.e., such that the sequence $x, \phi(x), \phi^2(x), \ldots$ has the fewest different states). In that case, either $\phi(x) > x$, but $\phi^2(x) < \phi(x)$ or vice versa; without loss of generality consider the former case. Denote $y = \phi(x)$; then the sequence $y, \phi(y), \phi^2(y), \ldots$ is monotone by construction of x. Consequently, $\{V_i(s)\}_{i \in N}^{s \in \{y, \phi(y), \phi^2(y), \ldots\}}$ satisfies increasing differences. By property 3 of Definition 3 applied to state y, for all $i \in M_y$, $V_i(\phi(y)) \geq V_i(y)$. This is true for all $i \in M_x$. However, property 2 of Definition 3, applied to state x, implies that, generically, at least for one $i \in M_x$, $V_i(y) > V_i(\phi(y))$. This contradiction completes the proof. ■

Proof of Theorem 11. Take an increasing sequence of sets of points, $S_1 \subset S_2 \subset S_3 \subset \cdots$, so that $\bigcup_{i=1}^{\infty} S_i$ is dense. For each S_i, take MVE ϕ_i. We know that ϕ_i is a monotone function on S_i; let us complement it to a monotone (not necessarily continuous) function on S which we denote by $\tilde{\phi}_i$ for each i. Since $\tilde{\phi}_i$ are monotone functions from a bounded set to a bounded set, there is a subsequence $\tilde{\phi}_{i_k}$ which converges to some $\tilde{\phi}$ pointwisely. (Indeed, we can pick a subsequence which converges on S_1, then a subsequence converging on S_2 etc; then use a diagonal process. After it ends, the set of points where convergence was not achieved is at most countable, so we can repeat the diagonal procedure.) To show that $\tilde{\phi}$ is a MVE, suppose not, then there are two points x and y such that y is preferred to $\tilde{\phi}(x)$ by all members of M_x. Here, we need to apply a continuity argument and say that it means that the same is true for some points in some S_i. But this would yield a contradiction. ■

Examples

Example 5 (Example where the limit state depends on the timing of shocks). There are two environments, E_1 and E_2, with the probability of transition $\pi(E_1, E_2) = 0.1$. There are two states A, B, and two players 1 and 2. In both environments, the decision-making rule is dictatorship of player 1 in state A and dictatorship of player 2 in state B. All transitions are feasible, and the discound factor is $\beta = 0.9$. Payoffs are given by

$$\begin{array}{ccc} E_1 & A & B \\ 1 & 5 & 20 \\ 2 & 20 & 30 \end{array}, \quad \begin{array}{ccc} E_2 & A & B \\ 1 & 30 & 20 \\ 2 & 20 & 30 \end{array}.$$

Then the mapping ϕ is given by $\phi^{E_1}(A,B) = (B,B)$; $\phi^{E_2}(A,B) = (A,B)$. Suppose that $s_0 = 1$. Then, if the shock arrives in period $t = 1$, the limit state is A, and if the shock arrives later, the limit state is B.

Example 6 ***(Example with single-peaked preferences and two MVE)*** There are three states A, B, C, and two players 1 and 2. The decision-making rule is unanimity in state A and dictatorship of player 2 in states B and C. Payoffs are given by

$$\begin{array}{cccc} & A & B & C \\ 1 & 2 & 25 & 20 \\ 2 & 1 & 20 & 25 \end{array}$$

Then ϕ_1 given by $\phi_1(A,B,C) = (B,C,C)$ and ϕ_2 given by $\phi_2(A,B,C) = (C,C,C)$ are both MVE when the discount factor is any $\beta \in [0,1)$

Example 7 ***(Continuation utilities need not satisfy single-peakedness)*** There are four states and three players, player 1 is the dictator in state A, player 2 is the dictator in state B, and player 3 is the dictator in states C and D. The payoffs are given by the following matrix:

$$\begin{array}{ccccc} & A & B & C & D \\ 1 & 20 & 30 & 90 & 30 \\ 2 & 5 & 20 & 85 & 90 \\ 3 & 5 & 25 & 92 & 99 \end{array}.$$

All payoffs are single-peaked. Suppose $\beta = 0.5$; then the unique equilibrium has $\phi(A) = C$, $\phi(B) = \phi(C) = \phi(D) = D$. Let us compute the continuation payoffs of player 1. We have: $V_1(A) = 40$, $V_1(B) = 30$, $V_1(C) = 50$, $V_1(D) = 30$; the continuation utility of player 1 is thus not single-peaked.

Example 8 ***(No MVE with infinite number of shocks)*** Below is an example with finite number of states and players and finite number of environments such that all assumptions, except for the assumption that the number of shocks is finite, are satisfied, but there is no Markov Voting Equilibrium in pure strategies.

There are three environments E_1, E_2, E_3, three states $A = 1, B = 2, C = 3$, and three players $1, 2, 3$. The history of environments follows a simple Markov chain; in fact, in each period the environment is drawn separately. More precisely,

$$\begin{aligned}
\pi(E_1) &: = \pi(E_1, E_1) = \pi(E_2, E_1) = \pi(E_3, E_1) = \frac{1}{2}; \\
\pi(E_2) &: = \pi(E_1, E_2) = \pi(E_2, E_2) = \pi(E_3, E_2) = \frac{2}{5}; \\
\pi(E_3) &: = \pi(E_1, E_2) = \pi(E_2, E_3) = \pi(E_3, E_3) = \frac{1}{10}.
\end{aligned}$$

The discount factor is $\frac{1}{2}$.

The following matrices describe stage payoffs, winning coalitions, and feasible transitions.

Environment E_1	*State* A	*State* B	*State* C
Winning coalition	*Dictatorship of Player* 1		
Feasible transitions	*to* A, B	*to* B	*to* C
Player 1	60	150	−800
Player 2	30	130	60
Player 3	−100	60	50

Environment E_2	*State* A	*State* B	*State* C
Winning coalition	*Dictatorship of Player* 2		
Feasible transitions	*to* A	*to* A, B	*to* C
Player 1	100	80	−800
Player 2	80	70	60
Player 3	−100	60	50

Environment E_3	*State* A	*State* B	*State* C
Winning coalition	*Dictatorship of Player* 3		
Feasible transitions	*to* A	*to* B, C	*to* C
Player 1	100	80	−800
Player 2	80	70	60
Player 3	−100	60	50

It is straightforward to see that Sincreasing differences holds; moreover, payoffs are single-peaked, and in each environment and each state, the set of quasi-median voters is a singleton.

The intuition behind the example is the following. The payoff matrices in environment E_2 and E_3 coincide, so "essentially", there are two equally likely environments E_1 and "$E_2 \cup E_3$". Both player 1 and 2 prefer state B when the environment is E_2 and state A when the environment is E_1; given the payoff matrix and the discount factor, player 1 would prefer to move from A to B when in E_1, and knowing this, player 2 would be willing to move to A when in E_2. However, there is a chance that the environment becomes E_3 rather than E_2, in which case a "maniac" player 3 will become able to move from state B (but not from A!) to state C; the reason for him to do so is that although he likes state B (in all environments), he strongly dislikes A, and thus if players 1 and 2 are expected to move between these states, player 3 would rather lock the society in state C, which is only slighly worse for him than B.

State C, however, is really hated by player 1, who would not risk the slightest chance of getting there. So, if player 3 is expeced to move to C when given such chance, player 1 would not move from A to B when the environment is E_1, because player 3 is only able to move to C from B. Now player 2, anticipating that if he decides to move from B to A when the environment is E_2, the society will end up in state A forever; this is something player 2 would like to avoid, because state A is very bad for him when the environment is E_1. In short, if player 3 is expected to move to C when given this chance, then the logic of the previous paragraph breaks down, and neither player 1 nor player 2 will be willing to move when they are in power. But in this case, player 3 is better off staying in state B even when given a chance to move to C, as he trades off staying in B forever versus staying in C forever. These considerations should prove that there is no MVE.

More formally, note that there are only eight candidate mappings to consider (some transitions are made infeasible precisely to simplify the argument; alternatively, we could allow any transitions and make player 1 the dictator in state A when the environment is E_3). We consider these eight mappings separately, and point out the deviation. Obviously, the only values of the transition mappings to be specified are $\phi_{E_1}(A)$, $\phi_{E_2}(B)$, and $\phi_{E_3}(B)$.

1. $\phi_{E_1}(A) = A$, $\phi_{E_2}(B) = A$, $\phi_{E_3}(B) = B$. Then $\phi'_{E_3}(B) = C$ is a profitable deviation.
2. $\phi_{E_1}(A) = B$, $\phi_{E_2}(B) = A$, $\phi_{E_3}(B) = B$. Then $\phi'_{E_3}(B) = C$ is a profitable deviation.
3. $\phi_{E_1}(A) = A$, $\phi_{E_2}(B) = B$, $\phi_{E_3}(B) = B$. Then $\phi'_{E_1}(A) = B$ is a profitable deviation.
4. $\phi_{E_1}(A) = B$, $\phi_{E_2}(B) = B$, $\phi_{E_3}(B) = B$. Then $\phi'_{E_2}(B) = A$ is a profitable deviation.
5. $\phi_{E_1}(A) = A$, $\phi_{E_2}(B) = A$, $\phi_{E_3}(B) = C$. Then $\phi'_{E_2}(B) = B$ is a profitable deviation.
6. $\phi_{E_1}(A) = B$, $\phi_{E_2}(B) = A$, $\phi_{E_3}(B) = C$. Then $\phi'_{E_1}(A) = A$ is a profitable deviation.
7. $\phi_{E_1}(A) = A$, $\phi_{E_2}(B) = B$, $\phi_{E_3}(B) = C$. Then $\phi'_{E_3}(B) = B$ is a profitable deviation.
8. $\phi_{E_1}(A) = B$, $\phi_{E_2}(B) = B$, $\phi_{E_3}(B) = C$. Then $\phi'_{E_3}(B) = B$ is a profitable deviation.

This proves that there is no MVE in pure strategies (i.e., in the sense of Definition 3).