Online Appendix for “Dynamic Incentive Accounts”

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B Variable Cost of Effort

This section extends the core model to allow a deterministically varying marginal cost of effort. In practice, this occurs if either the cost function or maximum effort level changes over time. For example, for a start-up firm, the CEO can undertake many actions to improve firm value (augmenting the maximum effort level) and effort is relatively productive (reducing the cost of effort).

We now allow for a time-varying maximum effort level $\overline{a}_t$ and cost of effort $g_t(\cdot)$. The slope of the contract in Theorem 1 (equations (11) and (12)) now becomes:

$$\theta_t = \begin{cases} \frac{g_t'(\bar{a}_t)}{1+\rho+\ldots\rho^{T-t}} \text{ for } t \leq L, \\ 0 \text{ for } t > L \end{cases} \tag{58}$$

if manipulation is impossible, and if manipulation is possible

$$\theta_t = \begin{cases} \theta_t = \frac{\Theta}{1+\rho+\ldots\rho^{T-t}}\rho^{-t} \text{ for } t \leq L+M, \\ 0 \text{ for } t > L+M \end{cases} \quad . \tag{59}$$

where $\Theta = \sup_{s \leq L}(\rho^s g_s'(\bar{a}_s))$.

We previously showed that imposing the NM constraint causes the contract's slope to rise over time; the speed of the rise depended only on the CEO's impatience ρ. With a non-constant target action, it depends on $\Theta = \sup_{s \leq L}(\rho^s g_s'(\bar{a}_s))$, the maximum discounted sensitivity during the CEO's working life. Let $s \leq L$ denote the period in which $\rho^s g_s'(\bar{a}_s)$ is highest. The CEO has an incentive to increase r_s at the expense of the signal in any t within M periods of s. Therefore, the sensitivity for all t within M periods of s must increase, to remove these incentives. However, this in turn has a knock-on effect: since the sensitivity for $t = s - M$ has now risen, the CEO has an incentive to increase r_{s-M} at the expense of r_{s-2M}, and so on. Therefore, the sensitivity at s forces upward the sensitivity in all periods $t \leq L+M$, even those more than M periods away from s, owing to the knock-on effects. This “resonance” explains why the contract in all periods $t \leq L+M$ depends on Θ in equation (59).

This dependence can be illustrated in a numerical example. We first set $T = 5$, $L = 3$, $\rho = 1$, $g_1'(\overline{a}_1) = g_2'(\overline{a}_2) = 1$ and $g_3'(\overline{a}_3) = 2$. If manipulation is impossible, the optimal contract is

$$
\begin{aligned}
\ln c_1 &= \frac{r_1}{5} + \kappa_1 \\
\ln c_2 &= \frac{r_1}{5} + \frac{r_2}{4} + \kappa_2 \\
\ln c_3 &= \frac{r_1}{5} + \frac{r_2}{4} + \frac{2}{3} r_3 + \kappa_3 \\
\ln c_4 &= \frac{r_1}{5} + \frac{r_2}{4} + \frac{2}{3} r_3 + \kappa_4 \\
\ln c_5 &= \frac{r_1}{5} + \frac{r_2}{4} + \frac{2}{3} r_3 + \kappa_5.
\end{aligned}
$$

Since the marginal cost of effort is high at $t = 3$, the contract sensitivity must be high at $t = 3$ to satisfy the EF condition. However, this now gives the CEO incentives to engage in manipulation if it were possible. If he manipulates r_2 downwards by 1 unit to augment r_3 by 1 unit, lifetime consumption falls by 1 unit and rises by 2 units. Therefore, the sensitivity of the contract at $t = 2$ must increase to remove these incentives. This increased sensitivity at $t = 2$ in turn augments the required sensitivity at $t = 1$, else the CEO would manipulate to reduce r_1 and increase r_2. Therefore, even though the maximum release lag M is 1 and so the CEO cannot directly manipulate r_1 to affect r_3, the high sensitivity at r_3 still affects the sensitivity at r_1 by changing the sensitivity at r_2. The new contract is given by:

$$
\begin{aligned}
\ln c_1 &= \frac{2}{5} r_1 + \kappa_1 \\
\ln c_2 &= \frac{2}{5} r_1 + \frac{r_2}{2} + \kappa_2 \\
\ln c_3 &= \frac{2}{5} r_1 + \frac{r_2}{2} + \frac{2}{3} r_3 + \kappa_3 \\
\ln c_4 &= \frac{2}{5} r_1 + \frac{r_2}{2} + \frac{2}{3} r_3 + r_4 + \kappa_4 \\
\ln c_5 &= \frac{2}{5} r_1 + \frac{r_2}{2} + \frac{2}{3} r_3 + r_4 + \kappa_5.
\end{aligned}
$$

C Analysis of Theorem 2

This section provides the analysis behind the comparative statics of the determinants of θ_t, discussed in the main paper shortly after Theorem 2. To study the impact of volatility on the contract, we parameterize the innovations by $\varepsilon_t = \sigma \varepsilon'_t$, where σ indicates volatility. We define the function:

$$
G(\theta, \gamma, \sigma) = \frac{\gamma - 1}{\gamma} \ln E\left[e^{-\gamma\theta\sigma\varepsilon'}\right] - \ln E\left[e^{(1-\gamma)\sigma\theta\varepsilon'}\right] \tag{60}
$$

in the domain $\theta \geq 0, \sigma \geq 0, \gamma \geq 1$. For instance, when ε' is a standard normal, $G(\theta, \gamma, \sigma) = \theta^2 \sigma^2 \frac{\gamma - 1}{2}$, and G is increasing in θ, γ, and σ.

We also define

$$H(\theta,\gamma,\sigma) = G(\theta,\gamma,\sigma) - \frac{\ln\rho + R}{\gamma} \tag{61}$$

If $\ln\rho + R$ is sufficiently small, then $H(\theta,\gamma,\sigma)$ is increasing in in θ,γ,σ.

Lemma 11 *Consider the domain $\theta \geq 0, \sigma \geq 0, \gamma \geq 1$, in the case where $\phi = 0$, $T = L$, and without the NM constraint. Suppose that $H(\theta,\gamma,\sigma)$ is increasing in its arguments in that domain. Then, $\theta_T = g'(\overline{a})$, and for $t < T$, θ_t is increasing in γ, in σ, and decreasing in ρ. If $H(\theta,\gamma,\sigma)$ is close enough to 0, then θ_t is increasing in t.*

The lemma means that the slope profile is increasing, and becomes flatter as γ and σ are higher. The intuition is thus: a higher γ, a higher σ, or a lower ρ, tend to decrease the relative importance of future consumptions $E\left[\rho^t c_t^{1-\gamma}\right]$. Hence, it is important to give a higher slope to the agent early on. By contrast, when γ is low, future consumptions are more important and so it is sufficient to give a lower slope early on.
Proof Using Theorem 2, simple calculations show, for $t \leq L$,

$$\begin{aligned}
\theta_t &= \frac{g'(\overline{a})}{\sum_{s=t}^{T} \rho^{s-t} \prod_{n=t+1}^{s} e^{-G(\theta_n,\gamma,\sigma) + \frac{1-\gamma}{\gamma}(R+\ln\rho)}} \\
&= \frac{g'(\overline{a})}{\sum_{s=t}^{T} \prod_{n=t+1}^{s} e^{-G(\theta_n,\gamma,\sigma) + \frac{1-\gamma}{\gamma}R + \frac{1}{\gamma}\ln\rho}} \\
\theta_t &= \frac{g'(\overline{a})}{\sum_{s=t}^{T} e^{-\sum_{n=t+1}^{s}(H(\theta_n,\gamma,\sigma)+R)}}
\end{aligned} \tag{62}$$

We have $\theta_T = g'(\overline{a})$. Proceeding by backward induction on t, starting at $t = T$, we see that θ_t is increasing in γ: this is because a higher γ increases $H(\theta_n,\gamma,\sigma)$ via the direct effect on H, and the effect on the future θ_n $(n > t)$, so it increases θ_t. The same reasoning holds for the comparative statics with respect to σ and ρ.

The last part of Lemma 11 comes from the fact that when $H \to 0$, $\theta_t \to \frac{g'(\overline{a})}{\sum_{s=t}^{T} e^{-R(t-s)}}$, which is increasing in t. ■

Another tractable case is the infinite horizon limit, where $T = L \to \infty$. Since the problem is stationary, θ_t is equal to a limit θ. From (62), this satisfies:

$$\theta = g'(\overline{a})\left(1 - e^{-H(\theta,\gamma,\sigma)-R}\right). \tag{63}$$

For instance, in the continuous-time, Gaussian noise limit,

$$\theta = g'(\overline{a})\left[\theta^2\sigma^2\frac{\gamma-1}{2} - \frac{\ln\rho + R}{\gamma} + R\right]. \tag{64}$$

It is economically clear that the lower root is the relevant one (for instance, it is increasing in the marginal cost of effort; it is also the root that is the limit of the finite-T slope). The slope of incentives (θ) is higher when the agent is more risk-averse (higher γ), there is more risk (higher σ), and the agent is less patient (lower ρ).

D The Optimality of No Manipulation

Section 4.3 proves that it is optimal for the principal to implement maximum effort in every period if the firm is large enough, in the case where manipulation is not possible. This section provides a potential microfoundation for the optimality of zero manipulation. We generalize the dividend expression (5) to:

$$D_\tau = (1-\mu) X \exp\left(\sum_{s=1}^{\tau} (\eta_s + a_s) - \sum_{s=1}^{\tau}\sum_{i=1}^{M} \lambda(m_{s,i})\right),$$

where $\mu = 0$ if the CEO engages in zero manipulation (i.e. $m_t = 0 \; \forall \; t$) and $\mu = \mu_* > 0$ if the probability that the CEO engages in manipulation is greater than zero. Thus, manipulation imposes a fixed cost on firm value: the expectation of even an infinitessimal amount of manipulation lowers firm value by a fixed amount μ_*. This technological assumption gives a tractable way to capture the fact that the possibility of manipulation leads to a step-change reduction in value (e.g. because monitoring is needed to verify accounts or scrutinize investment projects.) Note that the assumption of this cost μ allows us to dispense with the cost $\lambda(m)$ featured in the main paper.

Hence, the loss in expected firm value from allowing manipulation is bX with

$$b \equiv \mu_* e^{-R\tau} E_0 \left[\exp\left(\sum_{s=1}^{\tau} (\eta_s + \overline{a})\right)\right],$$

where X is baseline firm value without manipulation, while the benefit is at most A_0, the present value of the CEO's salary under the optimal contract which deters manipulation. Thus, if X is sufficiently large (if it is greater than A_0/b), no manipulation is optimal.

E Continuous Time

We now consider the continuous-time analog of the model. The CEO's utility is given by:

$$U = \begin{cases} E\left[\int_0^T \rho^t \frac{(c_t h(a_t))^{1-\gamma}-1}{1-\gamma} dt\right] & \text{if } \gamma \neq 1 \\ E\left[\int_0^T \rho^t (\ln c_t + \ln h(a_t))\, dt\right] & \text{if } \gamma = 1. \end{cases} \tag{65}$$

The firm's returns evolve according to:

$$dR_t = a_t dt + \sigma_t dZ_t$$

where Z_t is a Brownian motion, and the volatility process σ_t is deterministic. We normalize $r_0 = 0$ and the risk premium to zero, i.e. the expected rate of return on the stock is R in each period.

Proposition 1 *(Optimal contract, continuous time, log utility). The continuous-time limit of the optimal contract pays the CEO c_t at each instant, where c_t satisfies:*

$$\ln c_t = \int_0^t \theta_s dR_s + \kappa_t, \tag{66}$$

where θ_s and κ_t are deterministic functions. If manipulation is impossible, the slope θ_t is given by:

$$\theta_t = \begin{cases} \frac{g'(\bar{a})}{\int_t^T \rho^{\tau-s} ds} & \text{for } t \leq L \\ 0 & \text{for } t > L. \end{cases} \tag{67}$$

If manipulation is possible, θ_t is given by:

$$\theta_t = \begin{cases} \frac{g'(\bar{a})\rho^{-t}}{\int_t^T \rho^{\tau-s} ds} & \text{for } t \leq L + M \\ 0 & \text{for } t > L + M. \end{cases} \tag{68}$$

If private saving is impossible, the constant κ_t is given by

$$\kappa_t = (R + \ln \rho)\, t - \int_0^t \theta_s E\left[dR_s\right] - \zeta \int_0^t \frac{\theta_s^2 \sigma_s^2}{2} ds + \underline{\kappa}. \tag{69}$$

If private saving is possible, κ_t is given by

$$\kappa_t = (R + \ln \rho)\, t - \int_0^t \theta_s E\left[dR_s\right] + \zeta \int_0^t \frac{\theta_s^2 \sigma_s^2}{2} ds + \underline{\kappa}. \tag{70}$$

where $\underline{\kappa}$ ensures that the agent is at his reservation utility.

Proposition 2 *(Optimal contract, continuous time, general CRRA utility, with Private Savings constraint). Let σ_t denote the stock volatility. The optimal contract pays the CEO c_t at each instant, where c_t satisfies:*

$$\ln c_t = \int_0^t \theta_s dR_s + \kappa_t, \tag{71}$$

where θ_s and κ_t are deterministic functions. The continuous-time limit of the optimal contract is the following. If manipulation is impossible, the slope θ_t is given by:

$$\begin{aligned} \theta_t &= \frac{\rho^t e^{-(1-\gamma)g(\bar{a})} g'(\bar{a})}{\int_t^T \rho^s e^{-(1-\gamma)g(\bar{a}) + (1-\gamma)(\kappa_s - \kappa_t)} E_t\left[e^{(1-\gamma)\int_t^s \theta_\tau dR_\tau}\right] ds} && \text{for } t \leq L, \\ \theta_t &= 0 && \text{for } t > L. \end{aligned} \tag{72}$$

If manipulation is possible, θ_t is given by:

$$\theta_t = \frac{De^{(1-\gamma)(\kappa_{L+M}-\kappa_t)}E_t\left[e^{(1-\gamma)\int_t^{L+M}\theta_\tau dR_\tau}\right]}{\int_t^T \rho^s e^{-(1-\gamma)g(\bar{a})+(1-\gamma)(\kappa_s-\kappa_t)}E_t\left[e^{(1-\gamma)\int_t^s \theta_\tau dR_\tau}\right]ds} \qquad \textit{for } t \le L+M,$$

$$\theta_t = 0 \qquad \textit{for } t > L+M.$$

The value of κ_t is:

$$\gamma\kappa_t = (R+\ln\rho)t - (1-\gamma)g(\bar{a})\mathbf{1}_{t\ge L} - \gamma\int_0^t \theta_s \bar{a} ds + \frac{1}{2}\gamma^2\int_0^t \theta_s^2\sigma_s^2 ds + \underline{\kappa}, \tag{73}$$

where $\underline{\kappa}$ ensures that the agent is at his reservation utility, and D is the lowest constant such that:

$$De^{(1-\gamma)(\kappa_{L+M}-\kappa_t)}E_t\left[e^{(1-\gamma)\int_t^{L+M}\theta_\tau dR_\tau}\right] \ge \rho^t e^{-(1-\gamma)g(\bar{a})}g'(\bar{a}), \ \textit{for all } t \le L.$$

The implications of the optimal contract are the same as for discrete time, except that the rebalancing of the account is now continuous.

F Proofs of Lemmas

This section contains proofs of lemmas in the main paper.

Proof of Lemma 2 Let

$$P_s((b_t)_{t\le T}) = e^{\sum_{n=1}^{s-M} j_n(b_n) + \sum_{n=s-M+1}^{s} q_n^s(b_n)},$$

$$S_s((b_t)_{t\le T}) = \sum_{n=s}^{T} e^{\sum_{m=1}^{n-M} j_m(b_m) + \sum_{m=n-M+1}^{n} q_m^n(b_m)} = \sum_{n=s}^{T} P_n((b_t)_{t\le L}),$$

for any $s \le T$. For the rest of the proof, fix an argument sequence $(b_t)_{t\le T}$. We will evaluate all the functions at this sequence, and consequently economize on notation by dropping the argument of S_s, P_s, j_s and q_s^t.

Step 1: Derivatives. For unit vectors e_r^i and e_s^k, $r \ge s$, $i, k \le M+1$, consider the derivatives of the function I:

$$\frac{\partial I}{\partial e_s^k} = \sum_{n=s}^{s+M-1} \partial_k q_s^n P_n + \partial_k j_s S_{s+M},$$

$$\frac{\partial^2 I}{\partial e_r^i \partial e_s^k} = \sum_{n=r}^{s+M-1} \partial_k q_s^n \partial_i q_r^n P_n + \partial_k j_s \left(\sum_{n=\max\{r,s+M\}}^{r+M-1} \partial_i q_r^n P_n + \partial_i j_r S_{r+M} \right) +$$

$$+ \mathbf{1}_{r=s,i=k}\left[\sum_{n=s}^{s+M-1} \partial_k^2 q_s^n P_n + \partial_k^2 j_s S_{s+M}\right],$$

where we define $\partial_k f(x) = \frac{\partial}{\partial x_k} f(x)$ and $\partial_k^2 f(x) = \frac{\partial^2}{(\partial x_k)} f(x)$. Therefore, for a fixed vector $y = (y_t)_{t\leq T}$ the second derivative in the direction $y = (y_t)_{t\leq T}$ is:

$$
\begin{aligned}
\frac{\partial^2 I}{\partial y \partial y} =& \sum_{k,i=1}^{M+1} \sum_{s=1}^{T} \sum_{r=1}^{T} y_s^k y_r^i \frac{\partial^2 I}{\partial e_s^k \partial e_r^i} = \\
=& 2 \sum_{k,i=1}^{M+1} \sum_{s=1}^{T} \sum_{r\geq s} y_s^k y_r^i \left[\sum_{n=r}^{s+M-1} \partial_k q_s^n \partial_i q_r^n P_n + \partial_k j_s \left(\sum_{n=\max\{r,s+M\}}^{r+M-1} \partial_i q_r^n P_n + \partial_i j_r S_{r+M} \right) \right] \\
&+ \sum_{i=1}^{M+1} \sum_{s=1}^{T} y_s^{i2} \left[\sum_{n=s}^{s+M-1} \partial_i^2 q_s^n P_n + \partial_i^2 j_s S_{s+M} \right] =: W + V.
\end{aligned}
$$

Step 2: Bounding P_r and S_r. For any $s \leq T$ and $q \leq T - s$ we have:

$$
P_{s+q} = e^{\sum_{n=1}^{s+q} j_n + \sum_{n=s+q-M+1}^{s} q_n^s} \leq e^{M \sup q_t} e^{\sum_{n=1}^{s+q} j_n} \leq e^{M \sup q_t + q \sup j_t} e^{\sum_{n=1}^{s} j_n} \leq e^{q \sup j_t + M(\sup q_t - \inf q_t)} P_s,
$$

It follows that for $\psi = \frac{\sup j_s}{2}$ we have:

$$
\sum_{r\geq s} P_r e^{-\psi(r-s)} \leq C_1 P_s, \quad \sum_{s,r\geq s} P_r y_r^2 e^{\psi(r-s)} = \sum_r y_r^2 P_r \sum_{s\leq r} e^{\psi(r-s)} \leq C_2 \sum_s P_s y_s^2, \tag{74}
$$

where

$$
C_1 = e^{M(\sup q_t - \inf q_t)} \sum_{n=0}^{T} e^{n\psi}, \; C_2 = \sum_{n=0}^{T} e^{n\psi}. \tag{75}
$$

Moreover, since $S_{s+q} \leq e^{q \sup j_t + M(\sup q_t - \inf q_t)} S_s$, the inequalities for S_r analogous to (74) also hold.

Step 3: Bounding the derivatives. For any vector $z = (z_t)_{t\leq T}$, $z_t \in \mathbb{R}$, we have:

$$
\begin{aligned}
\sum_{s,r\geq s} z_s z_r P_r =& \sum_s z_s \sum_{r\geq s} \sqrt{P_r} z_r e^{\frac{\psi}{2}(r-s)} \sqrt{P_r} e^{-\frac{\psi}{2}(r-s)} \leq \sum_s z_s \left(\sum_{r\geq s} P_r z_r^2 e^{\psi(r-s)} \right)^{1/2} \left(\sum_{r\geq s} P_r e^{-\psi(r-s)} \right)^{1/2} \\
\leq& \sqrt{C_1} \sum_s z_s \sqrt{P_s} \left(\sum_{r\geq s} P_r z_r^2 e^{\psi(r-s)} \right)^{1/2} \leq \sqrt{C_1} \left(\sum_s z_s^2 P_s \right)^{1/2} \left(\sum_s \left(\sum_{r\geq s} P_r z_r^2 e^{\psi(r-s)} \right) \right)^{1/2} \\
\leq& \sqrt{C_1 C_2} \left(\sum_s z_s^2 P_s \right)^{1/2} \left(\sum_s P_s z_s^2 \right)^{1/2} = C \sum_s z_s^2 P_s,
\end{aligned}
$$

where the first and third inequalities follow from the Cauchy-Schwartz inequality, and C_1 and C_2 are as in (75). Similarly, we obtain $\sum_{s,r\geq s} z_s z_r S_r \leq C \sum_s z_s^2 S_s$. Therefore:

$$
\begin{aligned}
W &= 2\sum_{k,i=1}^{M+1}\sum_{s=1}^{T}\sum_{r\geq s} y_s^k y_r^i \left[\sum_{n=r}^{s+M-1} \partial_k q_s^n \partial_i q_r^n P_n + \partial_k j_s \left(\sum_{n=\max\{r,s+M\}}^{r+M-1} \partial_i q_r^n P_n + \partial_i j_r S_{r+M}\right)\right] \\
&\leq 2\sum_{k,i=1}^{M+1}\left\{\sum_{n=1}^{T} P_n \left[\sum_{s\geq n-M, r\geq s} y_s^k \partial_k q_s^n y_r^i \partial_i q_r^n\right] + \sum_{m=0}^{M-1}\sum_{s=1}^{T}\sum_{r\geq s}\left[y_s^k \partial_k q_s^{s+m} y_r^i \partial_i q_r^{r+m} P_{r+m}\right] + \right. \\
&\left. + \sum_{s=1}^{T}\sum_{r\geq s}\left[y_s^k \partial_k j_s y_r^i \partial_i j_r S_{r+M}\right]\right\} \\
&\leq 2(M+1)^2\left\{\sum_{n=1}^{T} P_n \left[\sum_{s\geq n-M, r\geq s} \max_i (y_s^i \partial_i q_s^n) \max_i (y_r^i \partial_i q_r^n)\right]\right. \\
&\left. + \sum_{m=0}^{M-1}\sum_{s=1}^{T}\sum_{r\geq s}\left[\max_i (y_s^i \partial_i q_s^{s+m}) \max_i (y_r^i \partial_i q_r^{r+m}) P_{r+m}\right] + \sum_{s=1}^{T}\sum_{r\geq s}\left[\max_i (y_s^i \partial_i j_s) \max_i (y_r^i \partial_i j_r) S_{r+M}\right]\right\} \\
&\leq 2(M+1)^2\left\{\sum_{n=1}^{T} P_n \left[\sum_{s\geq n-M} M \max_i (y_s^i \partial_i q_s^n)^2\right] + \sum_{m=0}^{M-1}\sum_{s=1}^{T}\left[C \max_i (y_s^i \partial_i q_s^{s+m})^2 P_{s+m}\right]\right. \\
&\left. + \sum_{s=1}^{T}\left[C \max_i (y_s^i \partial_i j_s)^2 S_{r+M}\right]\right\} \\
&\leq 2(M+1)^2 \sum_{s=1}^{T}\sum_{i=1}^{M+1} y_s^{i2}\left[\sum_{m=0}^{M-1} (M+C)(\partial_i q_s^{s+m})^2 P_{s+m} + C(\partial_i j_s)^2 S_{r+M}\right].
\end{aligned}
$$

Finally,

$$
\begin{aligned}
\frac{\partial^2 I}{\partial y \partial y} &= W + V \\
&\leq \sum_{s=1}^{T}\sum_{i=1}^{M+1} y_s^{i2}\left[\sum_{n=s}^{s+M-1} \left(2(M+C)(M+1)^2 \left(\partial_i q_s^n\right)^2 + \partial_i^2 q_s^n\right) P_n + \left(2C(M+1)^2 \left(\partial_i j_s\right)^2 + \partial_i^2 j_s\right) S_s\right],
\end{aligned}
$$

establishing the Lemma. ■

Proof of Lemma 3 To show that $I\left((m_t)_{t\leq L}, (x_t)_{t\leq L}\right)$ is jointly concave in leisure $(x_t)_{t\leq L}$ and manipulations $(m_t)_{t\le L}$, we use Lemma 2 with $b_t = (m_t, x_t)$ and:

$$j_s(m_s,x_s) = (\theta_s - \phi\theta_{s+1})\left[f(x_s) - \overline{a} + \sum_{i=1}^{M}(m_{s,i} - \lambda(m_{s,i}))\right] - \sum_{i=1}^{M}(\theta_{s+i} - \phi\theta_{s+i+1})m_{s,i} + \ln\rho, \tag{76}$$

$$q_s^t(m_s,x_s) = (\theta_s - \phi\theta_{s+1})\left[f(x_s) - \overline{a} + \sum_{i=1}^{M}(m_{s,i} - \lambda(m_{s,i}))\right] - \sum_{i=1}^{t-s}(\theta_{s+i} - \phi\theta_{s+i+1})m_{s,i} + \ln\rho,\ \ s < t$$

$$q_s^s(m_s,x_s) = \theta_s\left[f(x_s) - \overline{a} + \sum_{i=1}^{M}(m_{s,i} - \lambda(m_{s,i}))\right] + \ln\rho.$$

We have $\theta_t - \phi\theta_{t+1} \le D_0\left(\theta_s - \phi\theta_{s+1}\right)$ as long as $t > s$ and $|t-s| \le M$, for some $D_0 > 0$. Let λ be such that $\sup D_0 m - \lambda(m) \le D_1$, for some $D_1 > 0$, and m^* be such that $D_0 m - \lambda(m) \le 0$ for $m \ge m^*$. We can assume without loss of generality that the CEO chooses manipulations only within the interval $[-m^*, m^*]$, and so

$$C = e^{M(\sup q_s^t - \inf q_s^t)/2}\sum_{n=0}^{T} e^{n\sup j_t/2}$$

is finite. Finally, since

$$f'(x_s) = \frac{-1}{g'(f(x_s))},\quad f''(x_s) = \frac{-g''(f(x_s))}{g'^3(f(x_s))}\text{ and }\theta_s \le g'(\overline{a}),$$

the condition (50) is satisfied for $i=1$ if g has sufficiently high curvature. Moreover, since

$$\begin{aligned}
\frac{\partial}{\partial m_{s,i}}q_s^t &= (\theta_s - \phi\theta_{s+1})(1-\lambda'(m_{s,i})) - \mathbf{1}_{t<s+i}(\theta_{s+i} - \phi\theta_{s+i+1}),\\
\frac{\partial}{\partial m_{s,i}}j_s &= (\theta_s - \phi\theta_{s+1})(1-\lambda'(m_{s,i})\text{ and}\\
\frac{\partial^2}{(\partial m_{s,i})^2}q_s^t &= \frac{\partial^2}{(\partial m_{s,i})^2}j_s = -(\theta_s - \phi\theta_{s+1})\lambda''(m_{s,i}),
\end{aligned}$$

the condition (50) is satisfied for $i > 1$ if λ has sufficiently high curvature. ■

Proof of Lemma 4 We must verify condition (50) in Lemma 2 for j_s and q_s^t defined as:

$$j_s(m_s,x_s) = (\theta_s - \phi\theta_{s+1})\left[f(x_s) - \gamma\overline{a} + \sum_{i=1}^{M}(m_{s,i} - \lambda(m_{s,i}))\right] - \sum_{i=1}^{M}\theta_{s+i}m_{s,i} + D_s, \tag{77}$$

$$q_s^t(m_s,x_s) = (\theta_s - \phi\theta_{S+1})\left[f(x_s) - \gamma\overline{a} + \sum_{i=1}^{M}(m_{s,i} - \lambda(m_{s,i}))\right] - \sum_{i=1}^{t-s}\theta_{s+i}m_{s,i} + D_s,\ \ s \le t,$$

for $D_s = (1-\gamma)k_s + \ln E\left(e^{(1-\gamma)\theta_s e_s}\right) + \ln\rho$. The rest of the proof follows as in the $\gamma = 1$ case, with the derivatives of the f function being:

$$f'(x_s) = -D\frac{1}{x_s g'(f(x_s))},\ f''(x_s) = \frac{1}{x_s^2 g'^2(f(x_s))}\left(Dg'(f(x_s)) - D^2\frac{g''(f(x_s))}{g'(f(x_s))}\right),$$

for $D = \frac{\gamma}{1-\gamma} sign(1-\gamma)$. Consequently $I'((m_t)_{t\leq L}, (x_t)_{t\leq L})$ is pathwise concave and so $E\widetilde{U}_\eta$ is concave in the processes $(x_t)_{t\leq L}$ and $(m_t)_{t\leq L}$. ■

Proof of Lemma 5 Let $U_t(\boldsymbol{\eta}_t; \eta_t')$ be the CEO's continuation utility after history $\boldsymbol{\eta}_t$ if the agent reports $\boldsymbol{\eta}_{t-1}, \eta_t'$. (54) follows from the standard envelope conditions, i.e. $\frac{\partial}{\partial \eta_t'} U_t(\boldsymbol{\eta}_t; \eta_t')|_{\eta_t'=\eta_t} = 0$ together with:

$$U_t(\boldsymbol{\eta}_t; \eta_t') = U_t(\boldsymbol{\eta}_{t-1}, \eta_t') + g(a_t(\boldsymbol{\eta}_{t-1}, \eta_t')) - g(a_t(\boldsymbol{\eta}_{t-1}, \eta_t') + \eta_t' - \eta_t),\ \text{for } \gamma = 1,$$

$$U_t(\boldsymbol{\eta}_t; \eta_t') = U_t(\boldsymbol{\eta}_{t-1}, \eta_t') + \frac{y_t(\boldsymbol{\eta}_{t-1}, \eta_t')^{1-\gamma}\left[e^{-g(a_t(\boldsymbol{\eta}_{t-1},\eta_t')+\eta_t'-\eta_t)(1-\gamma)} - e^{-g(a_t(\boldsymbol{\eta}_{t-1},\eta_t'))(1-\gamma)}\right]}{1-\gamma}.\ \text{for } \gamma \neq 1.$$

The technical assumptions on $a_t(\boldsymbol{\eta}_{t-1}, \cdot)$ guarantee that $U_t(\boldsymbol{\eta}_{t-1}, \cdot)$ is absolutely continuous (for details see e.g. EG p. 49). $y_t(\boldsymbol{\eta}_t) > 0$ follows from the private savings constraint, since the marginal utility of consumption at zero is infinite. ■

Proof of Lemma 6 Note that if instead of $U_t^{\#}(\boldsymbol{\eta}_{t-1}, \cdot)$ and $\zeta(\boldsymbol{\eta}_{t-1}, \cdot)$ we solve for the functions $\overline{U_t^{\#}}(\boldsymbol{\eta}_{t-1}, \cdot)$ and $\overline{\zeta}(\boldsymbol{\eta}_{t-1}, \cdot)$ that satisfy $\overline{U_t^{\#}}(\boldsymbol{\eta}_{t-1}, \underline{\eta}) = U_t(\boldsymbol{\eta}_{t-1}, \underline{\eta})$ and

$$\overline{U_t^{\#}}(\boldsymbol{\eta}_{t-1}, \eta_t) = \overline{U_t^{\#}}(\boldsymbol{\eta}_{t-1}, \underline{\eta}) + \int_{\underline{\eta}}^{\eta_t} [\overline{\zeta}(\boldsymbol{\eta}_{t-1}, x) y_t(\boldsymbol{\eta}_{t-1}, x) e^{-g(\overline{a})}]^{1-\gamma} g'(\overline{a}) dx, \tag{78}$$

$$\overline{U_t^{\#}}(\boldsymbol{\eta}_{t-1}, \eta_t) - U_t(\boldsymbol{\eta}_{t-1}, \eta_t) = g(a_t(\boldsymbol{\eta}_{t-1}, \eta_t)) - g(\overline{a}) + \ln \overline{\zeta}(\boldsymbol{\eta}_{t-1}, \eta_t), for \gamma = 1.$$

$$\frac{\overline{U_t^{\#}}(\boldsymbol{\eta}_{t-1}, \eta_t)}{U_t(\boldsymbol{\eta}_{t-1}, \eta_t)} = \frac{[\overline{\zeta}(\boldsymbol{\eta}_{t-1}, \eta_t) y_t(\boldsymbol{\eta}_{t-1}, \eta_t) e^{-g(\overline{a})}]^{1-\gamma}}{[y_t(\boldsymbol{\eta}_{t-1}, \eta_t) e^{-g(a_t(\boldsymbol{\eta}_{t-1}, \eta_t))}]^{1-\gamma}},\ \text{for } \gamma \neq 1,$$

then we have $\zeta(\boldsymbol{\eta}_{t-1}, \eta_t) \leq \overline{\zeta}(\boldsymbol{\eta}_{t-1}, \eta_t)$ (and $\zeta(\boldsymbol{\eta}_{t-1}, \eta_t) = \overline{\zeta}(\boldsymbol{\eta}_{t-1}, \eta_t)$ when $t = L$). Therefore it will be sufficient to $E_{t-1}\left[\overline{\zeta}(\boldsymbol{\eta}_{t-1}, \eta_t)\right]$.

Since $\boldsymbol{\eta}_{t-1}$ is fixed, to economize on notation we write $U_t(\eta_t)$ instead of $U_t(\boldsymbol{\eta}_{t-1}, \eta_t)$ etc.

Case $\gamma \neq 1$. We have:

$$\overline{U_t^{\#}}(\eta_t) = \overline{U_t^{\#}}(\underline{\eta}) + \int_{\underline{\eta}}^{\eta_t} \frac{\overline{U_t^{\#}}(x)}{U_t(x)} \left[y_t(x) e^{-g(a_t(x))}\right]^{1-\gamma} g'(a_t(x)) \frac{g'(\overline{a})}{g'(a_t(x))} dx,$$

$$U_t(\eta_t) = \overline{U_t^{\#}}(\underline{\eta}) + \int_{\underline{\eta}}^{\eta_t} \left[y_t(x) e^{-g(a_t(x))}\right]^{1-\gamma} g'(a_t(x)) dx.$$

Therefore:

$$\left(\frac{\overline{U_t^{\#}}(\eta_t)}{U_t(\eta_t)}\right)' =$$

$$= \frac{\frac{\overline{U_t^{\#}}(\eta_t)}{U_t(\eta_t)}\left[y_t(\eta_t)e^{-g(a_t(\eta_t))}\right]^{1-\gamma} g'(a_t(\eta_t))\frac{g'(\overline{a})}{g'(a_t(\eta_t))}U_t(\eta_t) - \left[y_t(\eta_t)e^{-g(a_t(\eta_t))}\right]^{1-\gamma} g'(a_t(\eta_t))\overline{U_t^{\#}}(\eta_t)}{U_t(\eta_t)^2} =$$

$$= \frac{\overline{U_t^{\#}}(\eta_t)\left[y_t(\eta_t)e^{-g(a_t(\eta_t))}\right]^{1-\gamma} g'(a_t(\eta_t))\left[\frac{g'(\overline{a})}{g'(a_t(\eta_t))}-1\right]}{U_t(\eta_t)^2} \le \frac{\overline{U_t^{\#}}(\eta_t)}{U_t(\eta_t)}(1-\gamma)g'(\overline{a})\left[\frac{g'(\overline{a})}{g'(a_t(\eta_t))}-1\right] \text{ for } \gamma<1,$$

It follows that:

$$\frac{\overline{U_t^{\#}}(\eta_t)}{U_t(\eta_t)} \le e^{(1-\gamma)g'(\overline{a})\int_{\underline{\eta}}^{\eta_t}\left(\frac{g'(\overline{a})}{g'(a_t(x))}-1\right)dx} \le e^{(1-\gamma)\sup\frac{g'(\overline{a})}{f}E_{t-1}\left(\frac{g'(\overline{a})}{g'(a_t(x))}-1\right)} \le e^{(1-\gamma)g'(\overline{a})\sup\frac{g''}{fg'^2}E_{t-1}[\overline{a}-a_t(\boldsymbol{\eta}_t)]}, \text{ for } \gamma<1. \tag{79}$$

where the last inequality follows because $\frac{g'(\overline{a})}{g'(a)} = g'(\overline{a})\left[\frac{1}{g'(\overline{a})} + (\overline{a}-a)\frac{g''(x\overline{a}+(1-x)a)}{g'^2(x\overline{a}+(1-x)a)}\right]$ for some $x\in[0,1]$. For $\gamma>1$ we obtain the analogous chain with the inequality signs reversed. Thus,

$$\begin{aligned} E_{t-1}\left[\overline{\zeta}(\eta_t)\right] &= E_{t-1}\left[\left[\frac{\overline{U_t^{\#}}(\eta_t)}{U_t(\eta_t)}\right]^{\frac{1}{1-\gamma}} e^{[g(\overline{a})-g(a_t(\eta_t))](1-\gamma)}\right] \le \\ &\le e^{g'(\overline{a})\sup\frac{g''}{fg'^2}E_{t-1}[\overline{a}-a_t(\eta_t)]}E_{t-1}\left[e^{[g(\overline{a})-g(a_t(\eta_t))](1-\gamma)}\right] \le \\ &\le e^{g'(\overline{a})\sup\frac{g''}{fg'^2}E_{t-1}[\overline{a}-a_t(\eta_t)]}\left(1+\mathbf{1}_{\gamma<1}e^{g(\overline{a})-g(\underline{a})}(1-\gamma)g'(\overline{a})E_{t-1}\left[\overline{a}-a_t(\eta_t)\right]\right). \end{aligned} \tag{80}$$

Case $\gamma=1$. Comparing (54) and (78) we immediately obtain:

$$\ln\overline{\zeta}(\eta_t) = \int_{\underline{\eta}}^{\eta_t}\left(\frac{g'(\overline{a})}{g'(a_t(x))}-1\right)g'(a_t(x))dx + g(\overline{a}) - g(a_t(\eta_t)).$$

Using the analogous bounds as in (79) and (80) we obtain:

$$\begin{aligned} E_{t-1}\left[\overline{\zeta}(\eta_t)\right] &\le E_{t-1}\left[e^{g'(\overline{a})\int_{\underline{\eta}}^{\eta_t}\left(\frac{g'(\overline{a})}{g'(a_t(x))}-1\right)dx+g(\overline{a})-g(a_t(x))}\right] \le e^{g'(\overline{a})\sup\frac{g''}{fg'^2}E_{t-1}[\overline{a}-a_t(\eta_t)]}E_{t-1}\left[e^{g(\overline{a})-g(a_t(\eta_t))}\right] \le \\ &\le e^{g'(\overline{a})\sup\frac{g''}{fg'^2}E_{t-1}[\overline{a}-a_t(\eta_t)]}\left(1+e^{g(\overline{a})-g(\underline{a})}g'(\overline{a})E_{t-1}\left[\overline{a}-a_t(\eta_t)\right]\right). \end{aligned}$$

■

Proof of Lemma 7 Multiplying all payoffs by ζ results in all the continuation utilities $U_t(\boldsymbol{\eta}_t)$ and deviation continuation utilities $U_t(\boldsymbol{\eta}_t;\eta_t')$ multiplied by constant $\zeta^{1-\gamma}$ for $\gamma\neq1$, or having a constant $\ln\zeta\times\sum_{s=0}^{L-t}\rho^s$ added at time t, for $\gamma=1$, and so the local EF constraint is unaffected. This also results in the marginal utilities of current consumption multiplied by $\zeta^{-\gamma}$, and so the local PS constraint is also unaffected. ■

Proof of Lemma 8 We prove only the $\gamma \neq 1$ case. For the $\overline{\zeta}$ as in the proof of Lemma (6) we have:

$$E_{t-1}\left[\frac{mu_t^{\#}(\boldsymbol{\eta}_{t-1},\eta_t)}{mu_t(\boldsymbol{\eta}_{t-1},\eta_t)}\right] \geq E_{t-1}\left[\overline{\zeta}^{-\gamma}(\boldsymbol{\eta}_{t-1},\eta_{t-1}) \times e^{(1-\gamma)(g(a(\boldsymbol{\eta}_{t-1},\eta_t))-g(\overline{a}))}\right] =$$

$$= E_{t-1}\left[\left[\frac{\overline{U_t^{\#}}(\boldsymbol{\eta}_{t-1},\eta_t)}{U_t(\boldsymbol{\eta}_{t-1},\eta_t)}\right]^{\frac{-\gamma}{1-\gamma}} e^{-\gamma(1-\gamma)[g(\overline{a})-g(a_t(\boldsymbol{\eta}_{t-1},\eta_t))]} \times e^{(1-\gamma)(g(a_t(\boldsymbol{\eta}_{t-1},\eta_t))-g(\overline{a}))}\right] =$$

$$\geq e^{-\gamma g'(\overline{a})\sup\frac{g''}{fg'^2}E_{t-1}[\overline{a}-a_t(\boldsymbol{\eta}_t)]}E_{t-1}\left[e^{-(1+\gamma)(1-\gamma)[g(\overline{a})-g(a_t(\boldsymbol{\eta}_{t-1},\eta_t))]}\right] \geq$$

$$\geq e^{-\gamma g'(\overline{a})\sup\frac{g''}{fg'^2}E_{t-1}[\overline{a}-a_t(\boldsymbol{\eta}_t)]}\left(1-\mathbf{1}_{\gamma<1}e^{-(1+\gamma)(1-\gamma)[g(\overline{a})-g(\underline{a})]}g'(\overline{a})(1-\gamma)(1+\gamma)E_{t-1}\left[\overline{a}-a_t(\boldsymbol{\eta}_t)\right]\right).$$

■

Proof of Lemma 9 We prove only the $\gamma \neq 1$ case. From (54) it follows that for every η_t and η_t' :

$$e^{(\overline{\eta}-\underline{\eta})g'(\overline{a})} \times y_t^h(\boldsymbol{\eta}_{t-1},\eta_t)^{1-\gamma}e^{-(1-\gamma)g(a_t^h(\boldsymbol{\eta}_{t-1},\eta_t))} \geq y_t^h(\boldsymbol{\eta}_{t-1},\eta_t')^{1-\gamma}e^{-(1-\gamma)g(a_t^h(\boldsymbol{\eta}_{t-1},\eta_t'))},$$

and so for every η_t and η_t' :

$$y^h(\boldsymbol{\eta}_{t-1},\eta_t')^{-\gamma}e^{\gamma g(a_t^h(\boldsymbol{\eta}_{t-1},\eta_t'))} \geq e^{-\left|\frac{\gamma}{1-\gamma}\right|(\overline{\eta}-\underline{\eta})g'(\overline{a})} \times y_t^h(\boldsymbol{\eta}_{t-1},\eta_t)^{-\gamma}e^{\gamma g(a^h(\boldsymbol{\eta}_{t-1},\eta_t))},$$
$$E_{t-1}\left[mu_t^h(\boldsymbol{\eta}_{t-1},\eta_t)\right] \geq e^{-\left|\frac{\gamma}{1-\gamma}\right|(\overline{\eta}-\underline{\eta})g'(\overline{a})+g(\underline{a})-g(\overline{a})} \times \max_x mu_t^h(\boldsymbol{\eta}_{t-1},x).$$

It follows that for $D_2 = e^{\left|\frac{\gamma}{1-\gamma}\right|(\overline{\eta}-\underline{\eta})g'(\overline{a})+g(\overline{a})-g(\underline{a})}$,

$$\frac{E_{t-1}\left[mu_t^l(\boldsymbol{\eta}_{t-1},\eta_t)\right]}{E_{t-1}\left[mu_t^h(\boldsymbol{\eta}_{t-1},\eta_t)\right]} \geq \frac{E_{t-1}\left[mu_t^h(\boldsymbol{\eta}_{t-1},\eta_t)\right]\left(1-D_2\times\left(1-E_{t-1}\left[\frac{mu_t^l(\eta_{t-1},\eta_t)}{mu_t^h(\boldsymbol{\eta}_{t-1},\eta_t)}\right]\right)\right)}{E_{t-1}\left[mu_t^h(\boldsymbol{\eta}_{t-1},\eta_t)\right]} =$$
$$= 1-D_2\times\left(1-E_{t-1}\left[\frac{mu_t^l(\boldsymbol{\eta}_{t-1},\eta_t)}{mu_t^h(\boldsymbol{\eta}_{t-1},\eta_t)}\right]\right).$$

■

Proof of Lemma 10 Let Y^0 be the payoff scheme Y^x. For any n, $0 < n < L$, we construct the payoff scheme Y^n as follows. Start with the payoff scheme Y^{n-1}. After any history $\boldsymbol{\eta}_n$ multiply the payoffs at time n by $\zeta^{n,ps}(\boldsymbol{\eta}_n) > 1$ so that the PS constraint at history $\boldsymbol{\eta}_n$ is satisfied; then multiply the payoffs after any history $\boldsymbol{\eta}_m$, $m \geq n$ and $\boldsymbol{\eta}_{m|n} = \boldsymbol{\eta}_n$, by $\zeta^{n,pu}(\boldsymbol{\eta}_n) < 1$ so that the continuation utility at history $\boldsymbol{\eta}_n$ remains unchanged. After any history $\boldsymbol{\eta}_{n-1}$ multiply the payoffs at time $n-1$ by $\zeta^{n,ps}(\boldsymbol{\eta}_{n-1}) > 1$ so that the PS constraint at $\boldsymbol{\eta}_{n-1}$ is satisfied; then multiply the payoffs after any history $\boldsymbol{\eta}_m$, $m \geq n-1$ and $\boldsymbol{\eta}_{m|n-1} = \boldsymbol{\eta}_{n-1}$, by $\zeta^{n,pu}(\boldsymbol{\eta}_{n-1}) < 1$ so that the continuation utility at $\boldsymbol{\eta}_{n-1}$ remains unchanged. Follow this procedure until histories at time 1, and let Y^n be the resulting payoff scheme. One can inductively show that $\zeta^{n,pu}(\boldsymbol{\eta}_m)\times$

$\zeta^{n,ps}(\boldsymbol{\eta}_m) \geq 1$, $m \leq n$.

Let A^* always require the maximum effort. Lemma 7 yields that each contract (A^*, Y^n) satisfies the local EF constraint and also satisfies the local PS constraint up to round n. Let $Y^* = Y^{L-1}$. It remains to prove (57).

For any history $\boldsymbol{\eta}_L$ we have $y^*_L(\eta_L) = y^x_L(\eta_L) \times \prod_{m=1}^{L-1} \prod_{n=m}^{L-1} \zeta^{n,pu}(\boldsymbol{\eta}_{L|m}) \leq y^x_L(\eta_L)$ and so the condition (57) is satisfied.

For any history $\boldsymbol{\eta}_t$, $t < L$, we have, by construction above:

$$\frac{mu^*_t(\boldsymbol{\eta}_t)}{mu^x_t(\boldsymbol{\eta}_t)} = \left(\prod_{m=1}^{t} \prod_{n=m}^{L-1} \zeta^{n,pu}(\boldsymbol{\eta}_{t|m}) \times \prod_{n=t}^{L-1} \zeta^{n,ps}(\boldsymbol{\eta}_t) \right)^{-\gamma} \geq \left(\prod_{n=t}^{L-1} \zeta^{n,ps}(\boldsymbol{\eta}_t) \right)^{-\gamma}.$$

Moreover,

$$\begin{aligned} \zeta^{t,ps}(\boldsymbol{\eta}_t)^{-\gamma} &= \frac{E_t\left[mu^x_{t+1}(\boldsymbol{\eta}_{t-1}, \eta_t)\right]}{E_t\left[mu_{t+1}(\boldsymbol{\eta}_{t-1}, \eta_t)\right]} \geq \phi\left(E_t\left[\frac{mu^x_{t+1}(\boldsymbol{\eta}_{t-1}, \eta_t)}{mu_{t+1}(\boldsymbol{\eta}_{t-1}, \eta_t)} \right] \right) \geq \\ &\geq \phi\left(\psi\left(E_t\left[\overline{a} - a_{t+1}(\eta_{t+1})\right]\right)\right), \end{aligned}$$

where the first inequality follows from Lemma 9, and the second one from Lemma 8. By the same logic, for any n, $t < n \leq L-1$,

$$\begin{aligned} \zeta^{n,ps}(\boldsymbol{\eta}_t)^{-\gamma} &= \frac{E_t\left[mu^n_{t+1}(\boldsymbol{\eta}_t, \eta_{t+1})\right]}{E_t\left[mu^{n-1}_{t+1}(\boldsymbol{\eta}_t, \eta_{t+1})\right]} \geq \phi\left(E_t\left[\frac{mu^n_{t+1}(\boldsymbol{\eta}_t, \eta_{t+1})}{mu^{n-1}_{t+1}(\boldsymbol{\eta}_t, \eta_{t+1})} \right] \right) \geq \phi\left(E_t\left[\zeta^{n,ps}(\boldsymbol{\eta}_t, \eta_{t+1})^{-\gamma} \right] \right) \\ &= \phi\left(E_t\left[\frac{E_{t+1}\left[mu^n_{t+2}(\eta_t, \eta_{t+1}, \eta_{t+2})\right]}{E_{t+1}\left[mu^{n-1}_{t+2}(\boldsymbol{\eta}_t, \eta_{t+1}, \eta_{t+2})\right]} \right] \right) \geq \phi\left(E_t\left[\phi\left(E_{t+1}\left[\frac{mu^n_{t+2}(\boldsymbol{\eta}_t, \eta_{t+1}, \eta_{t+2})}{mu^{n-1}_{t+2}(\boldsymbol{\eta}_t, \eta_{t+1}, \eta_{t+2})} \right] \right) \right] \right) \\ &= \phi^2\left(E_t\left[\frac{mu^n_{t+2}(\boldsymbol{\eta}_t, \eta_{t+1}, \eta_{t+2})}{mu^{n-1}_{t+2}(\boldsymbol{\eta}_t, \eta_{t+1}, \eta_{t+2})} \right] \right) \geq \ldots \geq \phi^{n-t}\left(E_t\left[\frac{mu^n_n(\boldsymbol{\eta}_t, \eta_{t+1}, \ldots, \eta_n)}{mu^{n-1}_n(\boldsymbol{\eta}_t, \eta_{t+1}, \ldots, \eta_n)} \right] \right) \\ &\geq \phi^{n-t}\left(E_t\left[\zeta^{n,ps}(\boldsymbol{\eta}_t, \eta_{t+1}, \ldots, \eta_n)^{-\gamma} \right] \right) = \phi^{n-t}\left(E_t\left[\frac{E_n\left[mu^x_{n+1}(\boldsymbol{\eta}_t, \eta_{t+1}, \ldots, \eta_{n+1})\right]}{E_n\left[mu_{n+1}(\boldsymbol{\eta}_t, \eta_{t+1}, \ldots, \eta_{n+1})\right]} \right] \right) \\ &\geq \phi^{n-t+1}\left(E_t\left[\psi\left(E_n\left[\overline{a} - a_{n+1}(\boldsymbol{\eta}_t, \eta_{t+1}, \ldots, \eta_{n+1}) \right] \right) \right] \right). \end{aligned}$$

■