Technical Appendix

On the Unstable Relationship between Exchange Rates and Macroeconomic Fundamentals

Philippe Bacchetta and Eric van Wincoop

May 2009

In this Technical Appendix we cover two topics. First, we address how to implement in Gauss the expressions for the exchange rate and its derivative that are in the Appendix of the paper. Second, we derive the order components that are reported in section 2.5 of the paper.

I Numerical Implementation

For convenience we repeat equation (49) in the Appendix of the paper:

$$\begin{aligned}
s_t - s_{t-1} = \sum_{n=1}^{N} \left(\frac{\beta}{1-\rho\lambda} + (1-\lambda)(\beta_{nt}-\beta) \right) (f_{nt} - f_{n,t-1}) + \\
\left(\sum_{n=1}^{N} \left[\lambda \bar{\omega}^n + \bar{\theta}^n \right] (f_{nt} - f_{n,t-1}) + \lambda \sum_{n=1}^{N} \bar{h}_t^n + \tilde{b}\bar{b} \right) \mathbf{C}_t \boldsymbol{\omega}_t - \\
\left(\sum_{n=1}^{N} \bar{\theta}^n (f_{n,t-1} - f_{n,t-2}) + \lambda \sum_{n=1}^{N} \bar{f}_{t-1}^n + \tilde{b}\bar{b} \right) \mathbf{C}_{t-1} \boldsymbol{\omega}_{t-1} + \\
- \sum_{n=1}^{N} \frac{\rho\lambda\beta}{1-\rho\lambda} (f_{n,t-1} - f_{n,t-2}) + \lambda \sum_{n=1}^{N} \sum_{i=1}^{T} (f_{n,t-i} - f_{n,t-i-1}) \theta_{T-i+1} \epsilon_{n,t-T} + \\
(1-\lambda)(b_t - b_{t-1}) + \tilde{b}\rho_b^T (b_{t-T} - b_{t-T-1}) - \lambda (\nu_t - \nu_{t-1})
\end{aligned} \tag{1}$$

The difficulty in implementing this numerically is that the matrix $\mathbf{C}_t$ is very large, of size $(N+1)T$, which is 6000 based on our parameterization. In order to speed up the simulations in Gauss, we manipulate the matrices to obtain an expression for $\mathbf{C}_t \boldsymbol{\omega}_t$ that only involves matrices of size $T = 1000$. This is done as follows. We have

$$\mathbf{H}_t' \tilde{\mathbf{P}} = \sigma^2 \left(\mathbf{A}_{1t} ... \mathbf{A}_{Nt} \ \frac{\sigma_b^2}{\sigma^2} \mathbf{B} \right) \tag{2}$$

and

$$\mathbf{H}_t' \tilde{\mathbf{P}} \mathbf{H}_t = \sigma^2 \left(\sum_{n=1}^{N} \mathbf{A}_{nt} \mathbf{A}_{nt}' + \frac{\sigma_b^2}{\sigma^2} \mathbf{B}\mathbf{B}' \right) \equiv \sigma^2 \mathbf{D}_t \tag{3}$$

We also have

$$\tilde{\mathbf{P}}\mathbf{H}_t = \sigma^2 \begin{pmatrix} \mathbf{A}'_{1t} \\ \dots \\ \mathbf{A}'_{Nt} \\ \frac{\sigma_b^2}{\sigma^2}\mathbf{B}' \end{pmatrix} \tag{4}$$

so that

$$\mathbf{M}_t{=}\tilde{\mathbf{P}}\mathbf{H}_t \left[\mathbf{H}_t'\tilde{\mathbf{P}}\mathbf{H}_t\right]^{-1} = \begin{pmatrix} \mathbf{A}'_{1t}\mathbf{D}_t^{-1} \\ \dots \\ \mathbf{A}'_{Nt}\mathbf{D}_t^{-1} \\ \frac{\sigma_b^2}{\sigma^2}\mathbf{B}'\mathbf{D}_t^{-1} \end{pmatrix} \tag{5}$$

and

$$\mathbf{C}_t = \mathbf{M}_t\mathbf{H}_t' = \begin{pmatrix} \mathbf{A}'_{1t}\mathbf{D}_t^{-1}\mathbf{A}_{1t} & \dots & \mathbf{A}'_{1t}\mathbf{D}_t^{-1}\mathbf{A}_{Nt} & \mathbf{A}'_{1t}\mathbf{D}_t^{-1}\mathbf{B} \\ & \dots & \dots & \dots \\ \mathbf{A}'_{Nt}\mathbf{D}_t^{-1}\mathbf{A}_{1t} & \dots & \mathbf{A}'_{Nt}\mathbf{D}_t^{-1}\mathbf{A}_{Nt} & \mathbf{A}'_{Nt}\mathbf{D}_t^{-1}\mathbf{B} \\ \frac{\sigma_b^2}{\sigma^2}\mathbf{B}'\mathbf{D}_t^{-1}\mathbf{A}_{1t} & \dots & \frac{\sigma_b^2}{\sigma^2}\mathbf{B}'\mathbf{D}_t^{-1}\mathbf{A}_{Nt} & \frac{\sigma_b^2}{\sigma^2}\mathbf{B}'\mathbf{D}_t^{-1}\mathbf{B} \end{pmatrix} \tag{6}$$

This implies

$$\mathbf{C}_t\boldsymbol{\omega}_t = \begin{pmatrix} \sum_{i=1}^N \mathbf{A}'_{1t}\mathbf{D}_t^{-1}\mathbf{A}_{it}\boldsymbol{\xi}_{it} + \mathbf{A}'_{1t}\mathbf{D}_t^{-1}\mathbf{B}\mathbf{b}_t \\ \dots \\ \sum_{i=1}^N \mathbf{A}'_{Nt}\mathbf{D}_t^{-1}\mathbf{A}_{it}\boldsymbol{\xi}_{it} + \mathbf{A}'_{Nt}\mathbf{D}_t^{-1}\mathbf{B}\mathbf{b}_t \\ \frac{\sigma_b^2}{\sigma^2}\sum_{i=1}^N \mathbf{B}'\mathbf{D}_t^{-1}\mathbf{A}_{it}\boldsymbol{\xi}_{it} + \frac{\sigma_b^2}{\sigma^2}\mathbf{B}'\mathbf{D}_t^{-1}\mathbf{B}\mathbf{b}_t \end{pmatrix} \tag{7}$$

Therefore

$$\left(\sum_{n=1}^N \left[\lambda\bar{\omega}^n + \bar{\theta}^n\right](f_{nt} - f_{n,t-1}) + \lambda\sum_{n=1}^N \bar{h}_t^n + \tilde{b}\bar{b}\right)\mathbf{C}_t\boldsymbol{\omega}_t = \tag{8}$$
$$\left(\sum_{n=1}^N \left(\left[\lambda\hat{\omega} + \hat{\theta}\right](f_{nt} - f_{n,t-1}) + \lambda\hat{h}_{nt}\right)\mathbf{A}'_{nt} + \tilde{b}\frac{\sigma_b^2}{\sigma^2}\hat{b}\mathbf{B}'\right)\mathbf{D}_t^{-1}\left(\sum_{i=1}^N \mathbf{A}_{it}\xi_{it} + \mathbf{B}\mathbf{b}_t\right)$$

Using this result, the derivative of the exchange rate with respect to $\Delta f_{nt} = f_{nt} - f_{n,t-1}$ becomes

$$\partial s_t/\partial\Delta f_{nt} = \left(\frac{\beta}{1-\rho\lambda}\beta + (1-\lambda)(\beta_{nt} - \beta)\right) + \tag{9}$$
$$\frac{\partial\left(\sum_{n=1}^N \left[\lambda\bar{\omega}^n + \bar{\theta}^n\right](f_{nt} - f_{n,t-1}) + \lambda\sum_{n=1}^N \bar{h}_t^n + \tilde{b}\bar{b}\right)\mathbf{C}_t\boldsymbol{\omega}_t}{\partial\Delta f_{nt}}$$

where

$$\frac{\partial\left(\sum_{n=1}^{N}\left[\lambda\bar{\omega}^{n}+\bar{\theta}^{n}\right](f_{nt}-f_{n,t-1})+\lambda\sum_{n=1}^{N}\bar{h}_{t}^{n}+\tilde{b}\bar{b}\right)\mathbf{C}_{t}\boldsymbol{\omega}_{t}}{\partial\Delta f_{nt}}= \tag{10}$$

$$\left[\lambda\hat{\omega}+\hat{\theta}\right]\mathbf{A}_{nt}'\mathbf{D}_{t}^{-1}\left(\sum_{i=1}^{N}\mathbf{A}_{it}\xi_{it}+\mathbf{B}\mathbf{b}_{t}\right)+$$

$$\left(\left[\lambda\hat{\omega}+\hat{\theta}\right](f_{nt}-f_{n,t-1})+\lambda\hat{h}_{nt}\right)\frac{\partial\mathbf{A}_{nt}'}{\partial f_{nt}}\mathbf{D}_{t}^{-1}\left(\sum_{i=1}^{N}\mathbf{A}_{it}\xi_{it}+\mathbf{B}\mathbf{b}_{t}\right)+$$

$$\left(\sum_{n=1}^{N}\left(\left[\lambda\hat{\omega}+\hat{\theta}\right](f_{nt}-f_{n,t-1})+\lambda\hat{h}_{nt}\right)\mathbf{A}_{nt}'+\tilde{b}\frac{\sigma_{b}^{2}}{\sigma^{2}}\hat{b}\mathbf{B}'\right)\mathbf{D}_{t}^{-1}\frac{\partial\mathbf{A}_{nt}}{\partial f_{nt}}\xi_{nt}+$$

$$\left(\sum_{n=1}^{N}\left(\left[\lambda\hat{\omega}+\hat{\theta}\right](f_{nt}-f_{n,t-1})+\lambda\hat{h}_{nt}\right)\mathbf{A}_{nt}'+\tilde{b}\frac{\sigma_{b}^{2}}{\sigma^{2}}\hat{b}\mathbf{B}'\right)\frac{\partial\mathbf{D}_{t}^{-1}}{\partial\Delta f_{nt}}\left(\sum_{i=1}^{N}\mathbf{A}_{it}\xi_{it}+\mathbf{B}\mathbf{b}_{t}\right)$$

Using that for any matrix $\mathbf{D}$ we have $\mathbf{D}\mathbf{D}^{-1}=\mathbf{I}$, it follows that

$$\frac{\partial\mathbf{D}}{\partial x}\mathbf{D}^{-1}+\mathbf{D}\frac{\partial\mathbf{D}^{-1}}{\partial x}=0 \tag{11}$$

Therefore

$$\frac{\partial\mathbf{D}^{-1}}{\partial x}=-\mathbf{D}^{-1}\frac{\partial\mathbf{D}}{\partial x}\mathbf{D}^{-1} \tag{12}$$

Applying this here we have

$$\frac{\partial\mathbf{D}_{t}^{-1}}{\partial\Delta f_{nt}}=-\mathbf{D}_{t}^{-1}\left(\mathbf{A}_{nt}\frac{\partial\mathbf{A}_{nt}'}{\partial f_{nt}}+\frac{\partial\mathbf{A}_{nt}}{\partial\Delta f_{nt}}\mathbf{A}_{nt}'\right)\mathbf{D}_{t}^{-1} \tag{13}$$

Note that $\frac{\partial\mathbf{A}_{nt}}{\partial\Delta f_{nt}}$ is a matrix with the same dimension as $\mathbf{A}_{nt}$ with $(\theta_1,...,\theta_T)$ in the first row and otherwise zeros.

II Order Components

We now compute the zero, first and second-order components of the derivative of the exchange rate with respect to fundamentals. Our starting point is equation (19) in the text, which assumes $\rho_f=\rho_b=0$. For convenience we repeat the equation here:

$$\frac{\partial\Delta s_t}{\partial\Delta f_{nt}}=(1-\lambda)\beta_{nt}+\lambda E_t\beta_{nt}+\lambda\sum_{i=0}^{T}\Delta\mathbf{f}_{t-i}'\frac{\partial E_t\boldsymbol{\beta}_{t-i}}{\partial\Delta f_{nt}} \tag{14}$$

where

$$E_t\beta_{nt} = \beta + \bar{\boldsymbol{\theta}}_n \mathbf{C}_t \boldsymbol{\omega}_t \tag{15}$$

$$E_t\boldsymbol{\beta}_{t-i} = \hat{\boldsymbol{\beta}}_{t-i} + \hat{\boldsymbol{\theta}}_i \mathbf{C}_t \boldsymbol{\omega}_t \tag{16}$$

Here $\bar{\boldsymbol{\theta}}_n$ is a 1 by $(N+1)T$ vector that has $(\theta_1, .., \theta_T)$ in elements $(n-1)T+1$ to nT and zeros otherwise. $\hat{\boldsymbol{\theta}}_i$ is a N by $(N+1)T$ matrix. In row n $(n = 1, .., N)$ it has $(\theta_1, .., \theta_{T-i})$ in elements $(n-1)T+i+1$ to nT. Row n of $\hat{\boldsymbol{\beta}}_{t-i}$ is equal to

$$\beta + \sum_{j=T+1}^{i+T} \theta_{j-i}\epsilon_{n,t-j+1} \tag{17}$$

We are now in a position to compute the order components of the derivative of $\partial\Delta s_t/\partial\Delta f_{nt}$. The zero-order component is the level when all current and past shocks go to zero. The first-order component is linear in the shocks. The second-order component is equal to the product of two shocks, and so on. We will only go to up to the second-order components. There are three types of shocks: shocks to the fundamentals, to the parameters and to the noise.

First consider the zero-order component. It is immediate from the equations above that

$$\frac{\partial\Delta s_t}{\partial\Delta f_{nt}}(0) = \beta \tag{18}$$

To the zero-order, the derivative is simply equal to the steady state parameter β. In the absence of shocks, this would be the derivative.

We now compute the first-order component of each of the terms on the right hand side of (14). It is immediate that $\beta_{nt}(1) = \beta_{nt} - \beta$. Next consider the expectation of β_{nt}. We have

$$[E_t\beta_{nt}](1) = \bar{\boldsymbol{\theta}}_n \mathbf{C}(0) \boldsymbol{\omega}_t \tag{19}$$

We have

$$\mathbf{C}(0) = \mathbf{M}(0)\mathbf{H}(0)' = \tilde{\mathbf{P}}\mathbf{H}(0)[\mathbf{H}(0)'\tilde{\mathbf{P}}\mathbf{H}(0)]^{-1}\mathbf{H}(0)' \tag{20}$$

where $\tilde{\mathbf{P}}$ is redefined as

$$\tilde{\mathbf{P}} = \begin{pmatrix} \frac{\sigma^2}{\sigma_b^2}\mathbf{I}_{NT} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_T \end{pmatrix} \tag{21}$$

and

$$\mathbf{H}(0)' = [0, .., 0, \mathbf{B}] \tag{22}$$

Therefore

$$\tilde{\mathbf{P}}\mathbf{H}(0) = \begin{pmatrix} \frac{\sigma^2}{\sigma_b^2}\mathbf{I}_{NT} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_T \end{pmatrix} \begin{pmatrix} 0 \\ \mathbf{B}' \end{pmatrix} = \begin{pmatrix} 0 \\ \mathbf{B}' \end{pmatrix} \tag{23}$$

and

$$\mathbf{H}(0)'\tilde{\mathbf{P}}\mathbf{H}(0) = \mathbf{B}\mathbf{B}' \tag{24}$$

It follows that

$$\mathbf{C}(0) = \tilde{\mathbf{P}}\mathbf{H}(0)[\mathbf{H}(0)'\tilde{\mathbf{P}}\mathbf{H}(0)]^{-1}\mathbf{H}(0)' = \begin{pmatrix} 0 \\ \mathbf{B}' \end{pmatrix} (\mathbf{B}\mathbf{B}')^{-1}[0, .., 0, \mathbf{B}] = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_T \end{pmatrix} \tag{25}$$

Since the vector $\bar{\boldsymbol{\theta}}_n$ has zeros in the last T elements, it follows that $\bar{\boldsymbol{\theta}}_n\mathbf{C}(0)\boldsymbol{\omega}_t = 0$, so that

$$[E_t\beta_{nt}](1) = 0 \tag{26}$$

It is immediate that the first-order component of the last-term of (14) is zero, so that

$$\frac{\partial \Delta s_t}{\partial \Delta f_{nt}}(1) = (1-\lambda)(\beta_{nt} - \beta) \tag{27}$$

The weight on β_{nt} is therefore reduced by the factor $1-\lambda$ relative to the case where the time-varying parameters are known. This is because actual parameter innovations have no effect on the first-order component of the expectation of β_{nt}.

We finally consider the second-order component of each of the terms on the right hand side of (14). It is immediate that $\beta_{nt}(2) = 0$. We have

$$[E_t\beta_{nt}](2) = \bar{\theta}_n\mathbf{C}_t(1)\boldsymbol{\omega}_t \tag{28}$$

We therefore need to compute $\mathbf{C}_t(1)$. We have:

$$\mathbf{C}_t(1) = \mathbf{M}(0)\mathbf{H}_t(1)' + \mathbf{M}_t(1)\mathbf{H}(0)' \tag{29}$$

We have

$$\mathbf{M}(0) = \tilde{\mathbf{P}}\mathbf{H}(0)[\mathbf{H}(0)'\tilde{\mathbf{P}}\mathbf{H}(0)]^{-1} = \begin{pmatrix} 0 \\ \mathbf{B}' \end{pmatrix} (\mathbf{B}\mathbf{B}')^{-1} = \begin{pmatrix} 0 \\ \mathbf{B}^{-1} \end{pmatrix} \tag{30}$$

$$\mathbf{H}(0)' = [0, .., 0, \mathbf{B}] \tag{31}$$

$$\mathbf{H}_t(1)' = [\mathbf{A}_{1t}, .., \mathbf{A}_{Nt}, 0] \tag{32}$$

Therefore we only need to compute $\mathbf{M}_t(1)$ in order to evaluate (29). We have

$$\mathbf{M}_t(1) = \tilde{\mathbf{P}}\mathbf{H}_t(1)[\mathbf{H}(0)'\tilde{\mathbf{P}}\mathbf{H}(0)]^{-1} + \tilde{\mathbf{P}}\mathbf{H}(0)\left([\mathbf{H}_t'\tilde{\mathbf{P}}\mathbf{H}_t]^{-1}\right)(1) \tag{33}$$

The last term is zero. To see that, write $\mathbf{Z}_t = \mathbf{H}_t'\tilde{\mathbf{P}}\mathbf{H}_t$. We know that $\mathbf{Z}_t[\mathbf{Z}_t]^{-1} = I$, so that

$$\mathbf{Z}(0)\left([\mathbf{Z}_t]^{-1}\right)(1) + \mathbf{Z}_t(1)\mathbf{Z}(0)^{-1} = 0 \tag{34}$$

It follows that $([\mathbf{Z}_t]^{-1})(1) = 0$ when $\mathbf{Z}_t(1) = 0$. This is indeed the case here:

$$\mathbf{Z}_t(1) = \mathbf{H}_t(1)'\tilde{\mathbf{P}}\mathbf{H}(0) + \mathbf{H}(0)'\tilde{\mathbf{P}}\mathbf{H}_t(1) =$$

$$[\mathbf{A}_{1t},..,\mathbf{A}_{Nt},0]\begin{pmatrix} 0 \\ \mathbf{B}' \end{pmatrix} + [0,..,0,\mathbf{B}]\begin{pmatrix} \frac{\sigma^2}{\sigma_b^2}\mathbf{I}_{NT} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_T \end{pmatrix}\begin{pmatrix} \mathbf{A}_{1t}' \\ .. \\ \mathbf{A}_{Nt}' \\ 0 \end{pmatrix} = 0 \tag{35}$$

So we have

$$\begin{aligned}\mathbf{M}_t(1) &= \tilde{\mathbf{P}}\mathbf{H}_t(1)[\mathbf{H}(0)'\tilde{\mathbf{P}}\mathbf{H}(0)]^{-1} = \\ &\begin{pmatrix} \frac{\sigma^2}{\sigma_b^2}\mathbf{I}_{NT} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_T \end{pmatrix}\begin{pmatrix} \mathbf{A}_{1t}' \\ .. \\ \mathbf{A}_{Nt}' \\ 0 \end{pmatrix}(\mathbf{B}\mathbf{B}')^{-1} = \frac{\sigma^2}{\sigma_b^2}\begin{pmatrix} \mathbf{A}_{1t}'(\mathbf{B}\mathbf{B}')^{-1} \\ .. \\ \mathbf{A}_{Nt}'(\mathbf{B}\mathbf{B}')^{-1} \\ 0 \end{pmatrix}\end{aligned} \tag{36}$$

Therefore

$$\begin{aligned}\mathbf{C}_t(1) &= \begin{pmatrix} 0 \\ \mathbf{B}^{-1} \end{pmatrix}[\mathbf{A}_{1t},..,\mathbf{A}_{Nt},0] + \\ &\frac{\sigma^2}{\sigma_b^2}\begin{pmatrix} \mathbf{A}_{1t}'(\mathbf{B}\mathbf{B}')^{-1} \\ .. \\ \mathbf{A}_{Nt}'(\mathbf{B}\mathbf{B}')^{-1} \\ 0 \end{pmatrix}[0,..,\mathbf{B}] = \\ &\begin{pmatrix} 0 & .. & 0 & \frac{\sigma^2}{\sigma_b^2}(\mathbf{B}^{-1}\mathbf{A}_{1t})' \\ .. & .. & .. & .. \\ 0 & .. & 0 & \frac{\sigma^2}{\sigma_b^2}(\mathbf{B}^{-1}\mathbf{A}_{Nt})' \\ \mathbf{B}^{-1}\mathbf{A}_{1t} & .. & \mathbf{B}^{-1}\mathbf{A}_{Nt} & 0 \end{pmatrix}\end{aligned} \tag{37}$$

Therefore

$$[E_t\beta_{nt}](2) = \bar{\theta}_n \mathbf{C}_t(1)\boldsymbol{\omega}_t = \tag{38}$$

$$\bar{\boldsymbol{\theta}}_n \begin{pmatrix} 0 & .. & 0 & \frac{\sigma^2}{\sigma_b^2}(\mathbf{B}^{-1}\mathbf{A}_{1t})' \\ .. & .. & .. & .. \\ 0 & .. & 0 & \frac{\sigma^2}{\sigma_b^2}(\mathbf{B}^{-1}\mathbf{A}_{Nt})' \\ \mathbf{B}^{-1}\mathbf{A}_{1t} & .. & \mathbf{B}^{-1}\mathbf{A}_{Nt} & 0 \end{pmatrix} \boldsymbol{\omega}_t = \frac{\sigma^2}{\sigma_b^2}[\theta_1, .., \theta_T](\mathbf{B}^{-1}\mathbf{A}_{nt})'\tilde{\epsilon}_t^b$$

where

$$\tilde{\epsilon}_t^b = \begin{pmatrix} \epsilon_t^b \\ .. \\ \epsilon_{t-T+1}^b \end{pmatrix} \tag{39}$$

Taking the transpose, this can also be written as

$$[E_t\beta_{nt}](2) = \frac{\sigma^2}{\sigma_b^2}[\epsilon_t^b, .., \epsilon_{t-T+1}^b]\mathbf{B}^{-1}\mathbf{A}_{nt} \begin{pmatrix} \theta_1 \\ .. \\ \theta_T \end{pmatrix} \tag{40}$$

The term $\mathbf{B}^{-1}\mathbf{A}_{nt}$ is equal to

$$\mathbf{B}^{-1}\mathbf{A}_{nt} = \begin{pmatrix} 1 & -\rho_b & 0 & 0 & .. & 0 \\ 0 & 1 & -\rho_b & 0 & .. & 0 \\ .. & .. & .. & .. & .. & .. \\ 0 & .. & .. & 0 & 1 & -\rho_b \\ 0 & .. & .. & .. & 0 & 1 \end{pmatrix} \begin{bmatrix} \hat{f}_{nt}(1) & \hat{f}_{nt}(2) & ... & \hat{f}_{nt}(T) \\ 0 & \hat{f}_{n,t-1}(1) & ... & \hat{f}_{n,t-1}(T-1) \\ & ... & ... & ... \\ 0 & 0 & ... & \hat{f}_{n,t-T+1}(1) \end{bmatrix}$$

Multiplying these matrices gives

$$\begin{bmatrix} \hat{f}_{nt}(1) & \hat{f}_{nt}(2) - \rho_b\hat{f}_{n,t-1}(1) & \hat{f}_{nt}(3) - \rho_b\hat{f}_{n,t-1}(2) & .. & \hat{f}_{nt}(T) - \rho_b\hat{f}_{n,t-1}(T-1) \\ 0 & \hat{f}_{n,t-1}(1) & \hat{f}_{n,t-1}(2) - \rho_b\hat{f}_{n,t-2}(1) & .. & \hat{f}_{n,t-1}(T-1) - \rho_b\hat{f}_{n,t-2}(T-2) \\ & .. & .. & .. & .. \\ 0 & .. & 0 & \hat{f}_{n,t-T+2}(1) & \hat{f}_{n,t-T+2}(2) - \rho_b\hat{f}_{n,t-T+1}(1) \\ 0 & .. & .. & 0 & \hat{f}_{n,t-T+1}(1) \end{bmatrix}$$

We can then write

$$[E_t\beta_{nt}](2) = \frac{\sigma^2}{\sigma_b^2}[\epsilon_t^b, .., \epsilon_{t-T+1}^b]\mathbf{B}^{-1}\mathbf{A}_{nt} \begin{pmatrix} \theta_1 \\ .. \\ \theta_T \end{pmatrix} = \tag{41}$$

$$\frac{\sigma^2}{\sigma_b^2}\sum_{i=0}^{T-1}\left[\theta_{i+1}\hat{f}_{n,t-i}(1) + \sum_{j=i}^{T-2}\theta_{j+2}(\hat{f}_{n,t-i}(j-i+2) - \rho_b\hat{f}_{n,t-i-1}(j-i+1))\right]\epsilon_{t-i}^b$$

Use that

$$\hat{f}_{n,t-i}(j-i+2) = \hat{f}_{n,t-i-1}(j-i+1) + \theta_{j-i+2}\Delta f_{n,t-i} \tag{42}$$

Then

$$\begin{aligned}
&\frac{\sigma_b^2}{\sigma^2}[E_t\beta_{nt}](2) = \qquad (43)\\
&\sum_{i=0}^{T-1}\left[\theta_{i+1}\theta_1\Delta f_{n,t-i} + \sum_{j=i}^{T-2}\theta_{j+2}\theta_{j-i+2}\Delta f_{n,t-i} + (1-\rho_b)\sum_{j=i}^{T-2}\theta_{j+2}\hat{f}_{n,t-i-1}(j-i+1))\right]\epsilon_{t-i}^b = \\
&\sum_{i=0}^{T-1}\left[\sum_{j=i-1}^{T-2}\theta_{j+2}\theta_{j-i+2}\Delta f_{n,t-i} + (1-\rho_b)\sum_{j=i}^{T-2}\sum_{k=1}^{j-i+1}\theta_{j+2}\theta_{j-i+2-k}\Delta f_{n,t-i-k}\right]\epsilon_{t-i}^b
\end{aligned}$$

Using that

$$\begin{aligned}
&\sum_{j=i}^{T-2}\sum_{k=1}^{j-i+1}\theta_{j+2}\theta_{j-i+2-k}\Delta f_{n,t-i-k} = \sum_{k=1}^{T-i-1}\left(\sum_{j=i+k-1}^{T-2}\theta_{j+2}\theta_{j-i+2-k}\right)\Delta f_{n,t-i-k} = \\
&\sum_{k=i+1}^{T-1}\left(\sum_{j=k-1}^{T-2}\theta_{j+2}\theta_{j+2-k}\right)\Delta f_{n,t-k} \qquad (44)
\end{aligned}$$

we get

$$\begin{aligned}
\frac{\sigma_b^2}{\sigma^2}[E_t\beta_{nt}](2) &= \sum_{i=0}^{T-1}\sum_{k=i}^{T-1}\sum_{j=k-1}^{T-2}\theta_{j+2}\theta_{j+2-k}\delta_{ik}\Delta f_{n,t-k}\epsilon_{t-i}^b = \\
&\sum_{i=0}^{T-1}\sum_{k=i}^{T-1}\sum_{j=1}^{T-k}\theta_j\theta_{j+k}\delta_{ik}\Delta f_{n,t-k}\epsilon_{t-i}^b \qquad (45)
\end{aligned}$$

where $\delta_{ik} = 1$ for $k = i$ and $\delta_{ik} = 1-\rho_b$ for $k > i$.

Finally, the second-order component of the last term in (14) is

$$\lambda\sum_{i=0}^{T}\Delta\mathbf{f}'_{t-i}\hat{\boldsymbol{\theta}}_i\frac{\partial\mathbf{C}_t(1)}{\partial\Delta f_{nt}}\boldsymbol{\omega}_t \tag{46}$$

We know that

$$\mathbf{C}_t(1) = \begin{pmatrix} 0 & .. & 0 & \frac{\sigma^2}{\sigma_b^2}(\mathbf{B}^{-1}\mathbf{A}_{1t})' \\ .. & .. & .. & .. \\ 0 & .. & 0 & \frac{\sigma^2}{\sigma_b^2}(\mathbf{B}^{-1}\mathbf{A}_{Nt})' \\ \mathbf{B}^{-1}\mathbf{A}_{1t} & .. & \mathbf{B}^{-1}\mathbf{A}_{Nt} & 0 \end{pmatrix} \tag{47}$$

Therefore

$$\frac{\partial \mathbf{C}_t(1)}{\partial \Delta f_{nt}} = \begin{pmatrix} 0 & .. & 0 & \frac{\sigma^2}{\sigma_b^2}(\mathbf{B}^{-1}\partial \mathbf{A}_{1t}/\partial \Delta f_{nt})' \\ .. & .. & .. & .. \\ 0 & .. & 0 & \frac{\sigma^2}{\sigma_b^2}(\mathbf{B}^{-1}\partial \mathbf{A}_{Nt}/\partial \Delta f_{nt})' \\ \mathbf{B}^{-1}\partial \mathbf{A}_{1t}/\partial \Delta f_{nt} & .. & \mathbf{B}^{-1}\partial \mathbf{A}_{Nt}/\partial \Delta f_{nt} & 0 \end{pmatrix} \quad (48)$$

We have $\partial \mathbf{A}_{it}/\partial \Delta f_{nt} = 0$ for $i \neq n$ and

$$\partial \mathbf{A}_{nt}/\partial \Delta f_{nt} = \begin{bmatrix} \theta_1 & ... & \theta_T \\ 0 & ... & 0 \\ ... & ... & ... \\ 0 & ... & 0 \end{bmatrix} \quad (49)$$

Multiplication by $\mathbf{B}^{-1}$ leaves this derivative unchanged:$\mathbf{B}^{-1}\partial \mathbf{A}_{nt}/\partial \Delta f_{nt} = \partial \mathbf{A}_{nt}/\partial \Delta f_{nt}$

It is now easily seen that $\hat{\theta}_i \frac{\partial \mathbf{C}_t(1)}{\partial \Delta f_{nt}} \boldsymbol{\omega}_t$ is an N by 1 vector that has zeros in all rows but row n. In row n it is equal to

$$\frac{\sigma^2}{\sigma_b^2} \left(\sum_{j=1}^{T-i} \theta_j \theta_{j+i} \right) \epsilon_t^b \quad (50)$$

Therefore (46) becomes

$$\frac{\sigma^2}{\sigma_b^2} \lambda \sum_{i=0}^{T} \left(\sum_{j=1}^{T-i} \theta_j \theta_{j+i} \right) \Delta f_{n,t-i} \epsilon_t^b \quad (51)$$

To summarize, we have

$$\frac{\partial \Delta s_t}{\partial \Delta f_{nt}}(2) = \frac{\sigma^2}{\sigma_b^2} \lambda \sum_{i=0}^{T-1} \sum_{k=i}^{T-1} \sum_{j=1}^{T-k} \theta_j \theta_{j+k} \delta_{ik} \Delta f_{n,t-k} \epsilon_{t-i}^b + \frac{\sigma^2}{\sigma_b^2} \lambda \sum_{i=0}^{T} \left(\sum_{j=1}^{T-i} \theta_j \theta_{j+i} \right) \Delta f_{n,t-i} \epsilon_t^b$$

$$\frac{\partial \Delta s_t}{\partial \Delta f_{nt}}(1) = (1-\lambda)(\beta_{nt} - \beta)$$

$$\frac{\partial \Delta s_t}{\partial \Delta f_{nt}}(0) = \beta$$