10 Appendix to "Inflation Determination with Taylor Rules: A Critical Review"

The Appendix to "Identification with Taylor Rules", collects the algebra for the three-equation model, including determinacy conditions, used in both papers. This Appendix only includes three issues special to this paper.

10.1 More budget constraints

This section presents a somewhat more careful discussion of budget constraints in the model of section 3.1.

The household flow budget constraint states that bonds sold + income + transfers = consumption + taxes + bonds bought,

$$B_{t-1}(t)+P_t\left(Y_t\right)=P_t\left(C_t+T_t\right)+E_t\left(Q_{t,t+1}\right)B_t(t+1).$$

The household also faces a constraint that the present value of terminal wealth is zero.

$$\lim_{j\to\infty}E_t\left[Q_{t,t+j}B_{t+j-1}(t+j)\right]=0.$$

This latter "transversality condition" can be weakened to a bound on borrowing, since the consumer will never choose to overaccumulate wealth. In finite economies, you can't die with debts, and this is the limit of that condition for infinite-period economies. The condition also prevents consumers from arbitrage between a sequence of spot markets and markets for infinitely-lived contingent claims. Iterating forward, these two conditions are equivalent to the present value budget constraint,

$$E_t\left(\sum_{j=0}^{\infty}Q_{t,t+j}P_{t+j}C_{t+j}\right)=B_{t-1}(t)+E_t\left(\sum_{j=0}^{\infty}Q_{t,t+j}P_{t+j}\left(Y_{t+j}-T_{t+j}\right).\right)$$

In real terms, these constraints are

$$\frac{B_{t-1}(t)}{P_t}+Y_t+G_t=C_t+T_t+E_t\left(m_{t+1}\frac{B_t(t+1)}{P_{t+1}}\right)$$

$$\lim_{j\to\infty}E_t\left[m_{t,t+j}\frac{B_{t+j-1}(t+j)}{P_{t+j}}\right]=0.$$

and hence

$$E_t\sum_{j=0}^{\infty}m_{t,t+j}C_{t+j}=\frac{B_{t-1}(t)}{P_t}+E_t\sum_{j=0}^{\infty}m_{t,t+j}\left(Y_{t+j}-T_{t+j}\right)$$

The government faces a flow identity, taxes plus bonds sold = spending plus bonds redeemed,

$$E_t\left(Q_{t,t+1}\right)B_t(t+1)+P_tT_t=B_t(t+1)$$

or, in real terms

$$E_t\left(m_{t,t+1}\frac{B_t(t+1)}{P_{t+1}}\right)+T_t=\frac{B_t(t+1)}{P_t}.$$

It is tempting to iterate this forward as well and derive a government "intertemporal budget constraint." However, the government does *not* face a transversality condition. This fact is easiest to see in a finite economy. As a matter of budget constraint, we do not let agents die with debts. However, suppose agents developed a utility for government debt; they decide it makes nice wallpaper and are willing to hold it in the last period rather than cash it in and consume the proceeds. They can do this, and nothing in the government budget constraint should rule this out. The statement that government debt is zero at the end of time is an equilibrium condition, deriving from the fact that consumers without such utility will choose not to hold it. Thus, the government can, as a matter of constraint, make policy plans that, at off-equilibrium prices, would violate the consumer's budget constraint.

Thus, Equation (11) is an equilibrium condition that derives from the consumer's present value budget constraint, equilibrium $C = Y$, and the transversality condition for the consumer's choice to be an optimum. It is not a "government budget constraint." A "budget constraint" limits the demands an agent can announce at off-equilibrium prices, and there is nothing that stops the government from announcing plans that violate this equilibrium condition at off-equilibrium prices. Cochrane (2005) gives an extended discussion of this point.

10.2 A model with money

This section gives a brief self-contained description of the monetary model from section 4.4. Utility is

$$E\sum_t \beta^t u(C_t, M_t/P_t).$$

Money M_t is acquired during period t, and provides utility benefits during period t, and then must be held overnight from period t to period $t+1$. Thus, the flow budget constraint is

$$B_{t-1}(t)+M_{t-1}+P_tY_t=P_tC_t+P_tT_t+\frac{1}{1+i_t}B_t(t+1)+M_{t+1}.$$

The first order condition (8) is

$$\beta\frac{U_c(C_{t+1},M_{t+1}/P_{t+1})}{U_c(C_t,M_t/P_t)}=m_{t+1}.$$

Thus, or directly by consuming less today, buying a bond, and consuming more tomorrow,

$$1=(1+i_t)\,\beta E_t\left[\frac{U_c(C_{t+1},M_{t+1}/P_{t+1})}{U_c(C_t,M_t/P_t)}\frac{1}{\Pi_{t+1}}\right],$$

which in perfect foresight becomes (21). To find the second first-order condition, hold one more bond and one less dollar overnight. The bond costs $1/(1+i)$, so this change gains

$1-1/(1+i)=i/(1+i)$ dollars for consumption, but loses the utility benefits of money. Thus, the first order condition is

$$\frac{i_t}{1+i_t}U_c(C_t, M_t/P_t) = U_m(C_t, M_t/P_t) \tag{44}$$

The last equation can be solved for a "money demand" equation, $M_t/P_t = L(C_t, i_t)$ and in equilibrium, $M_t/P_t = L(Y, i_t)$.

10.3 The three equation model with $E_t\pi_{t+2}$

Again, the model is

$$\begin{aligned} y_t &= E_t y_{t+1} - \sigma r_t \\ i_t &= r_t + E_t \pi_{t+1} \\ \pi_t &= \beta E_t \pi_{t+1} + \gamma y_t \end{aligned}$$

We can eliminate r, leading to two equations

$$\begin{aligned} E_t y_{t+1} &= y_t + \sigma\left(i_t - E_t\pi_{t+1}\right) \\ \beta E_t\pi_{t+1} &= \pi_t - \gamma y_t \end{aligned}$$

Consider a Taylor rule of the form.

$$i_t = \phi_{\pi,0}\pi_t + \phi_{\pi,1}E_t\pi_{t+1} + \phi_{\pi 2}E_t\pi_{t+2} + \phi_{y,0}y_t + \phi_{y,1}E_t y_{t+1}$$

Substituting, we have

$$\beta E_t\pi_{t+2} = E_t\pi_{t+1} - \gamma E_t y_{t+1}$$

$$i_t = \phi_{\pi,0}\pi_t + \phi_{\pi,1}E_t\pi_{t+1} + \phi_{\pi 2}/\beta\left(E_t\pi_{t+1} - \gamma E_t y_{t+1}\right) + \phi_{y,0}y_t + \phi_{y,1}E_t y_{t+1}$$

$$i_t = \phi_{\pi,0}\pi_t + \left(\phi_{\pi,1} + \phi_{\pi 2}/\beta\right)E_t\pi_{t+1} + \left(\phi_{y,1} - \phi_{\pi 2}\gamma/\beta\right)E_t y_{t+1} + \phi_{y,0}y_t$$

$$E_t y_{t+1} = y_t + \sigma\left[\phi_{\pi,0}\pi_t + \left(\phi_{\pi,1} + \phi_{\pi 2}/\beta - 1\right)E_t\pi_{t+1} + \left(\phi_{y,1} - \phi_{\pi 2}\gamma/\beta\right)E_t y_{t+1} + \phi_{y,0}y_t\right]$$

$$\left[1 - \sigma\phi_{y,1} + \sigma\phi_{\pi 2}\gamma/\beta\right]E_t y_{t+1} = \left(1 + \sigma\phi_{y,0}\right)y_t + \sigma\phi_{\pi,0}\pi_t + \sigma\left(\phi_{\pi,1} + \phi_{\pi 2}/\beta - 1\right)E_t\pi_{t+1}$$

$$\begin{bmatrix} 1 - \sigma\phi_{y,1} + \sigma\phi_{\pi 2}\gamma/\beta & \sigma\left(1 - \phi_{\pi,1} - \phi_{\pi 2}/\beta\right) \\ 0 & \beta \end{bmatrix}\begin{bmatrix} E_t y_{t+1} \\ E_t\pi_{t+1} \end{bmatrix} = \begin{bmatrix} 1 + \sigma\phi_{y,0} & \sigma\phi_{\pi,0} \\ -\gamma & 1 \end{bmatrix}\begin{bmatrix} y_t \\ \pi_t \end{bmatrix}$$

$$\begin{bmatrix} E_t y_{t+1} \\ E_t\pi_{t+1} \end{bmatrix} = \frac{1}{\beta}\begin{bmatrix} \frac{\beta}{1-\sigma\phi_{y,1}+\sigma\phi_{\pi 2}\gamma/\beta} & -\sigma\frac{1-\phi_{\pi,1}-\phi_{\pi 2}}{1-\sigma\phi_{y,1}+\sigma\phi_{\pi 2}\gamma/\beta} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 + \sigma\phi_{y,0} & \sigma\phi_{\pi,0} \\ -\gamma & 1 \end{bmatrix}\begin{bmatrix} y_t \\ \pi_t \end{bmatrix}$$

$$\begin{bmatrix} E_t y_{t+1} \\ E_t\pi_{t+1} \end{bmatrix} = \frac{1}{\beta}\begin{bmatrix} \frac{\beta\left(1+\sigma\phi_{y,0}\right)+\sigma\gamma\left(1-\phi_{\pi,1}-\phi_{\pi 2}/\beta\right)}{1-\sigma\phi_{y,1}+\sigma\phi_{\pi 2}\gamma/\beta} & \frac{\beta\sigma\phi_{\pi,0}-\sigma\left(1-\phi_{\pi,1}-\phi_{\pi 2}/\beta\right)}{1-\sigma\phi_{y,1}+\sigma\phi_{\pi 2}\gamma/\beta} \\ -\gamma & 1 \end{bmatrix}\begin{bmatrix} y_t \\ \pi_t \end{bmatrix}$$

The eigenvalues of the matrix solve

$$\left\{\frac{\beta + \sigma\beta\phi_{y,0} - \sigma\gamma\left(\phi_{\pi 2}/\beta + \phi_{\pi,1} - 1\right)}{1 + \sigma\left(\gamma\phi_{\pi 2}/\beta - \phi_{y,1}\right)} - \lambda\right\}\{1 - \lambda\} + \frac{\sigma\gamma\left(\beta\phi_{\pi,0} + \phi_{\pi 2}/\beta + \phi_{\pi,1} - 1\right)}{1 + \sigma\left(\gamma\phi_{\pi 2}/\beta - \phi_{y,1}\right)} = 0$$

$$
\begin{aligned}
0 &= \lambda^2 - \left(\frac{\beta + \sigma\beta\phi_{y,0} - \sigma\gamma\left(\phi_{\pi 2}/\beta + \phi_{\pi,1} - 1\right)}{1 + \sigma\left(\gamma\phi_{\pi 2}/\beta - \phi_{y,1}\right)} + 1\right)\lambda \\
&\quad + \frac{\beta + \sigma\beta\phi_{y,0} - \sigma\gamma\left(\phi_{\pi 2}/\beta + \phi_{\pi,1} - 1\right)}{1 + \sigma\left(\gamma\phi_{\pi 2}/\beta - \phi_{y,1}\right)} + \frac{\sigma\gamma\left(\beta\phi_{\pi,0} + \phi_{\pi 2}/\beta + \phi_{\pi,1} - 1\right)}{1 + \sigma\left(\gamma\phi_{\pi 2}/\beta - \phi_{y,1}\right)}
\end{aligned}
$$

$$
\begin{aligned}
0 &= \lambda^2 - \left(\frac{\beta + \sigma\beta\phi_{y,0} - \sigma\gamma\left(\phi_{\pi 2}/\beta + \phi_{\pi,1} - 1\right)}{1 + \sigma\left(\gamma\phi_{\pi 2}/\beta - \phi_{y,1}\right)} + 1\right)\lambda \\
&\quad + \frac{\beta + \sigma\beta\phi_{y,0} - \sigma\gamma\left(\phi_{\pi 2}/\beta + \phi_{\pi,1} - 1\right) + \sigma\gamma\left(\beta\phi_{\pi,0} + \phi_{\pi 2}/\beta + \phi_{\pi,1} - 1\right)}{1 + \sigma\left(\gamma\phi_{\pi 2}/\beta - \phi_{y,1}\right)}
\end{aligned}
$$

$$
\lambda^2 - \left(1 + \frac{\beta\left(1 + \sigma\phi_{y,0}\right) - \sigma\gamma\left(\phi_{\pi 2}/\beta + \phi_{\pi,1} - 1\right)}{1 + \sigma\left(\gamma\phi_{\pi 2}/\beta - \phi_{y,1}\right)}\right)\lambda + \frac{\beta + \sigma\beta\left(\phi_{y,0} + \gamma\phi_{\pi,0}\right)}{1 + \sigma\left(\gamma\phi_{\pi 2}/\beta - \phi_{y,1}\right)} = 0
$$

Thus, including the leading $1/\beta$ term, the eigenvalues are

$$
\begin{aligned}
\lambda &= \frac{1}{2\beta}\left(1 + \frac{\beta\left(1 + \sigma\phi_{y,0}\right) - \sigma\gamma\left(\phi_{\pi 2}/\beta + \phi_{\pi,1} - 1\right)}{1 + \sigma\left(\gamma\phi_{\pi 2}/\beta - \phi_{y,1}\right)}\right) \\
&\quad \pm \frac{1}{2\beta}\sqrt{\left(1 + \frac{\beta\left(1 + \sigma\phi_{y,0}\right) - \sigma\gamma\left(\phi_{\pi 2}/\beta + \phi_{\pi,1} - 1\right)}{1 + \sigma\left(\gamma\phi_{\pi 2}/\beta - \phi_{y,1}\right)}\right)^2 - 4\beta\frac{1 + \sigma\left(\phi_{y,0} + \gamma\phi_{\pi,0}\right)}{1 + \sigma\left(\gamma\phi_{\pi 2} - \phi_{y,1}\right)}}
\end{aligned}
$$

$$
\begin{aligned}
\lambda &= \frac{1}{2\beta}\left(\frac{1 + \beta\left(1 + \sigma\phi_{y,0}\right) - \sigma\phi_{y,1} - \sigma\gamma\left(\phi_{\pi,1} - 1\right)}{1 + \sigma\left(\gamma\phi_{\pi 2}/\beta - \phi_{y,1}\right)}\right) \\
&\quad \pm \frac{1}{2\beta}\sqrt{\left(\frac{1 + \beta\left(1 + \sigma\phi_{y,0}\right) - \sigma\phi_{y,1} - \sigma\gamma\left(\phi_{\pi,1} - 1\right)}{1 + \sigma\left(\gamma\phi_{\pi 2}/\beta - \phi_{y,1}\right)}\right)^2 - 4\beta\frac{1 + \sigma\left(\phi_{y,0} + \gamma\phi_{\pi,0}\right)}{1 + \sigma\left(\gamma\phi_{\pi 2}/\beta - \phi_{y,1}\right)}} \\
\lambda &= \frac{1}{2}\left(\frac{1 + \beta\left(1 + \sigma\phi_{y,0}\right) - \sigma\phi_{y,1} - \sigma\gamma\left(\phi_{\pi,1} - 1\right)}{\beta + \sigma\left(\gamma\phi_{\pi 2} - \beta\phi_{y,1}\right)}\right) \\
&\quad \pm \frac{1}{2}\sqrt{\left(\frac{1 + \beta\left(1 + \sigma\phi_{y,0}\right) - \sigma\phi_{y,1} - \sigma\gamma\left(\phi_{\pi,1} - 1\right)}{\beta + \sigma\left(\gamma\phi_{\pi 2} - \beta\phi_{y,1}\right)}\right)^2 - 4\frac{1 + \sigma\left(\phi_{y,0} + \gamma\phi_{\pi,0}\right)}{\beta + \sigma\left(\gamma\phi_{\pi 2} - \beta\phi_{y,1}\right)}}
\end{aligned}
$$

$$
\begin{aligned}
\lambda &= \frac{1}{2\left(\beta + \sigma\left(\gamma\phi_{\pi 2} - \beta\phi_{y,1}\right)\right)} \times \left\{1 + \beta\left(1 + \sigma\phi_{y,0}\right) - \sigma\phi_{y,1} - \sigma\gamma\left(\phi_{\pi,1} - 1\right)\right. \\
&\quad \left. \pm \sqrt{\left[1 + \beta\left(1 + \sigma\phi_{y,0}\right) - \sigma\phi_{y,1} - \sigma\gamma\left(\phi_{\pi,1} - 1\right)\right]^2 - 4\left[\beta + \sigma\left(\gamma\phi_{\pi 2} - \beta\phi_{y,1}\right)\right]\left[1 + \sigma\left(\phi_{y,0} + \gamma\phi_{\pi,0}\right)\right]}\right\}
\end{aligned} \tag{45}
$$