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Directed Search for Equilibrium Wage-Tenure Contracts

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February 2007

Abstract

I analyze the equilibrium in a labor market where firms offer wage-tenure contracts to direct the search of employed and unemployed workers. Each applicant observes all offers and there is no coordination among individuals. Workers' applications (as well as firms' recruiting decisions) are optimal. This optimality requires the equilibrium to be formulated differently from the that in the literature of undirected search. I provide such a formulation and show that the equilibrium exists. In the equilibrium, individuals explicitly tradeoff between an offer and the matching rate at that offer. This tradeoff yields a unique offer which is optimal for each worker to apply, and applicants are separated endogenously according to their current values. Despite such uniqueness and separation, there is a non-degenerate and continuous wage distribution of employed workers in the stationary equilibrium. The density of this distribution is increasing at low wages and decreasing at high wages. In all equilibrium contracts, wages increase with tenure, which results in quit rates to decrease with tenure. Moreover, the model makes novel predictions about individuals' job-to-job transition and comparative statics.

Keywords: Directed search, On-the-Job, Wage-tenure contracts.

* I have received useful comments from the participants of the workshops at University of Pennsylvania, Rochester and Yale. Financial support from the Bank of Canada Fellowship and from the Social Sciences and Humanities Research Council of Canada is gratefully acknowledged. The opinion expressed in the paper is my own and it does not reflect the view of the Bank of Canada. All errors are mine.

1. Introduction

Directed search is a matching process in which an individual can use his offer to affect his matching rate. The objective of this paper is to study the equilibrium in a labor market where firms offer wage-tenure contracts to direct workers' search. A wage-tenure contract is a time profile of wages which describes how a worker's wage will evolve with tenure. All firms post contracts before workers apply and each applicant observes all offers. Employed workers continue to search on the job for better contracts elsewhere. I characterize the equilibrium and establish its existence. The equilibrium yields novel predictions about job-to-job transitions, the wage distribution, and effects of unemployment policy.

To see why directed search is interesting to study, it is useful to contrast it with the large literature on undirected search developed from Diamond (1982), Mortensen (1982), and Pissarides (1990). There are two classes of models in this literature. In one class, as in the three pioneering works, prices (wages) are a result of bargaining after individuals are matched. In the other class, some individuals post prices but the searching individuals do not know who posted what prices (e.g., Burdett and Mortensen, 1998, and Burdett and Coles, 2003). In both classes of models, search is undirected because prices play no role, ex ante, to direct workers to particular matches.

Although undirected search is a useful way to model search, it misses some important features of labor market search. First, some search is directed rather than completely random. For example, searching workers often have information about wages from job advertisement, word of mouth, or referrals. This is particularly true for workers who search on the job. Second, it has been a long tradition in economics to treat prices as a useful mechanism to direct the allocation of resources, ex ante. By abandoning this role of prices, the literature of undirected search generates an array of market inefficiencies. The corrective policy depends on arbitrary details of matching and price determination processes (see Hosios, 1990). Directed search can eliminate most of these inefficiencies. Third, undirected search models generate wage dispersion that is sensitive to the assumption on how many wages a worker knows before search. Before the application, a worker is assumed to know either no wage quotes as in the three pioneering works, or one wage (the worker's current wage) as in the on-the-job search model of Burdett and Mortensen (1998). If each applicant knows two or more wages, instead, then wage dispersion disappears in those models. This sensitivity reduces the potency of undirected search as an explanation for wage dispersion. Directed search eliminates this sensitivity.

During the last fifteen years or so, a literature has grown to analyze directed search.

Peters (1984, 1991) and Montgomery (1991) provide two of the earliest formulations. Examples of the further exploration include Moen (1997), Acemoglu and Shimer (1999a,b), Julien, et al. (2000), Burdett, et al. (2001), Shi (2001, 2002), Coles and Eeckhout (2003), Galenianos and Kircher (2005), and Delacroix and Shi (2006). They have shown that an equilibrium with directed search and its efficiency properties are significantly different from those with undirected search.

This literature has not yet introduced wage-tenure contracts; instead, it has assumed that each firm posts a single fixed wage for the entire duration of the worker's employment in the firm. Moreover, only one model in this literature (i.e., Delacroix and Shi, 2006) has incorporated on-the-job search. Without wage-tenure contracts, the literature of directed search is unable to explain the empirical regularities that wages rise and quit rates fall with tenure (e.g., Farber, 1999). Without on-the-job search, a model cannot make predictions on job-to-job transitions which constitute a large part of the flow of workers in the data. Given the appealing features of directed search discussed above, there is an urgency to bridge these gaps between directed search theory and the data.

It is a challenging task to characterize a directed search equilibrium with contracts. To appreciate the challenge, let me compare the task with the one in undirected search, which is accomplished by Burdett and Coles (2003, termed as BC henceforth). With undirected search, one does not need to formulate workers' application decisions, because workers are assumed to send their applications randomly to a pool of recruiting firms. With directed search, in contrast, each worker's application must be optimal. In this decision, a worker makes the optimal tradeoff between the level of an offer and the likelihood of obtaining the offer. Similarly, each firm understands that it can raise the offer to entice more workers to apply to it. To describe this tradeoff, I need two new objects in addition to the set of optimal contracts. One is the employment rate function, which describes how the rate at which an applicant gets a particular offer varies with the offer. The other is the hiring rate function, which describes how the rate at which a recruiting firm successfully hires a worker varies with the offer. These functions are equilibrium objects: They must be consistent with the aggregation of individuals' optimal choices, and the hiring rate must ensure that all equilibrium offers earn the same expected profit to a firm. A challenge in establishing existence of the equilibrium is to show that these functions exist.

I formulate the equilibrium in an environment where all matches have the same productivity, and then establish the existence of the equilibrium. In equilibrium, the hiring rate associated with an offer is indeed an increasing function of the offer, and the employment rate is a decreasing function of the offer. Thus, the tradeoff between an offer and the matching rate is meaningful. A striking feature of the procedure of determining the equilibrium is that the matching rate functions, equilibrium contracts and value functions can all be determined first without any reference to the distributions of offers and workers. This feature makes the model suitable for business cycle research, as I will elaborate in the concluding section.

On wage-tenure contracts, the equilibrium extends several realistic properties from the BC model of undirected search to directed search. First, wages increase and quit rates fall with tenure. Second, all equilibrium contracts are sections of a baseline contract. In the baseline contract, wages start at the lowest equilibrium level and then increase with tenure. Any other equilibrium contract that starts at a different wage is identical to the remaining section of the baseline contract from that wage level onward. Third, wage-tenure contracts and on-the-job search generate wage dispersion among workers, even though all matches have the same productivity.

Beyond these similarities, the equilibrium with directed search has little resemblance to the one with undirected search. One main difference is the predictions about individuals' job-to-job transitions and wage mobility. With directed search, the tradeoff between an offer and the matching rate induces each worker to apply to a unique offer. Moreover, this target offer is an increasing function of the applicant's current state, i.e., the value generated by the worker's current contract or unemployment benefit. As workers choose to separate themselves this way in the application process, wage mobility is endogenously limited by workers' current states despite the fact that there is no difference in productivity across matches. Such limited wage mobility seems realistic (see Buchinsky and Hunt, 1999). In contrast, undirected search models assume that job applications are random and exogenous (e.g., BC, 2003, and Burdett and Mortensen, 1998). There, a worker can receive an offer that lies anywhere in the support of the wage distribution, and a worker can transit to any wage that lies above his current wage.

The second difference from undirected search lies in the wage distribution. Not only does wage dispersion survive when all applicants observe all offers before they apply, the distribution also has a more realistic shape here. The density function of the distribution of employed workers over wages is increasing at low wages and decreasing at high wages in the current model. This non-monotonic shape of the density function is an empirical regularity (see Kiefer and Neumann, 1993), but it is not the prediction of an undirected search model with homogeneous matches. Instead, an increasing density function of employed wages is *necessary* to support an equilibrium when search is undirected and matches are homogeneous. To modify this unrealistic prediction, the literature has introduced heterogeneity across matches (e.g., van den Berg and Ridder, 1998). It is important to know that directed search can generate the non-monotonic wage density without such heterogeneity.

The third difference lies in the effects of unemployment policy. In the current model, an increase in unemployment benefits in the first-order stochastic dominance has no effect on the set of equilibrium contracts or individual workers' job-to-job transition rates, although it affects the shape of the wage distribution. If search were undirected, however, such an increase in unemployment benefits would increase the slope of the wage-tenure profile and increase the transition rate from low wages to high wages, as well as affecting wage distributions. Similarly, an increase in the arrival rate of offers to unemployed workers has much smaller effects on equilibrium contracts and job-to-job transitions in the current model than in undirected search models. These differences reflect the feature described earlier that matching rates and optimal contracts in the current model can be determined first without any reference to the distributions of wages and workers.

Now, turn to the literature of directed search. To this literature, the main contribution of this paper is to incorporate wage-tenure contracts and on-the-job search, as discussed earlier. Here, a contrast to Delacroix and Shi (2006) is necessary, because that paper also examines directed search on the job and establishes the result of limited wage mobility. However, that paper does not analyze wage-tenure contracts; instead, firms are assumed to offer only fixed wages. By incorporating wage-tenure contracts, the current paper not only explains empirical regularities, but also simplifies the characterization of the equilibrium. In Delacroix and Shi, the equilibrium wage structure is a wage ladder, the discreteness of which makes the analysis of the equilibrium quite messy. Wage-tenure contracts fill in those gaps between wages and induce a continuous wage distribution, as any initial gap between two offers will eventually be filled in by the increasing wage profile. Moreover, the characterization of the equilibrium here is much more general than in Delacroix and Shi.

Another contribution of the current model to the literature of directed search is to generate a continuous distribution of wages among homogeneous matches. In a similar setting, the literature of directed search generates only a finite number, or even a singleton, of equilibrium wages. In the current model, the main cause of a continuous distribution of wages is optimal wage-tenure contracts. Because such contracts specify wages as an increasing function of tenure, they enable a worker who got jobs earlier to earn more than a worker who got jobs later, even if the two workers are employed under the same contract. Another cause of wage dispersion is search on the job, because workers who got jobs earlier apply to higher wages than do workers who did not get jobs.

To emphasize the differences between directed and undirected search, I maintain four assumptions imposed by BC. First, workers are risk averse; second, the capital market is imperfect so that workers cannot borrow against their future income. These two assumptions are important for generating the wage-tenure relationship, as discussed by BC. Third, a firm does not respond to the worker's outside offers. How reasonable this assumption is clearly varies across different types of markets.[1] In any case, the assumption is commonly imposed in the literature, and it enables me to compare the results clearly with those in BC. For a model of undirected search without this assumption, see Postel-Vinay and Robin (2002). Finally, I assume that the productivity of a match is public information and deterministic. For private information or learning about productivity, see Jovanovic (1979), Harris and Holmstrom (1982), and Moscarini (2005). Such productivity differences between matches or over time are clearly important for wage dynamics and turnover, but abstracting from them enables me to have the clearest exploration of search frictions.

2. The Model

Consider a labor market that lasts forever in continuous time. There is a unit measure of risk averse workers whose utility function is $u(w)$, where w is income. Workers do not have access to the financial market to borrow against their future wage income, and so the lower bound on wages is 0. All workers have the same productivity: when employed, each worker produces a flow of output, $y > 0$. When unemployed, a worker enjoys a flow of utility, $u(b)$, which is derived from leisure and other benefits in unemployment. I will refer to b simply as the unemployment benefit.

To simplify the analysis, I assume that unemployment benefits are distributed in an interval, rather than being concentrated on a discrete set or degenerate.[2] More precisely, at (and only at) the time when a worker enters unemployment, he draws a value of b from the interval $[\underline{b}, \bar{b}]$ according to a continuous distribution H, where $0 < \underline{b} < \bar{b}$. Let the density function be $h(b) = H'(b)$ and assume that h is differentiable. To simplify the analysis further, I set $\bar{b} = \bar{w}$, where $\bar{w}$ is the highest wage specified later.

[1] In most occupations, workers who obtain outside offers often choose to quit rather than asking their current employers to match the offers. However, in some occupations such as economists and professional athletes, it is common for employers to match workers' outside offers. In these occupations, a main motivation for matching offers might be the competition for workers' ability. Because the current model abstracts from all heterogeneity in productivity across matches, the assumption of not responding to outside offers may not be unreasonable.

[2] The assumption that b is continuously distributed is made purely for technical convenience. Most of the analysis and results will remain unchanged even if the distribution of b is degenerate. In particular, a continuous wage dispersion will exist even if b is the same for all unemployed workers. See the end of section 4 for more discussion.

All workers face the process of death at a Poisson rate $\sigma \in (0, \infty)$. Dead workers are replaced with newborns who enter the labor market through unemployment and who draw unemployment benefits according to the distribution H. In addition to endogenous quits and exogenous death σ, there is no other cause of job separation. This modelling approach is convenient because it implies that a worker has no incentive to save if wages increase with tenure. I normalize the rate of time preference to zero. However, the probability of death generates effective discounting on the future.

Assumption 1. *The utility function has the following properties:* $0 < u'(w) < \infty$ *and* $-\infty < u''(w) < 0$ *for all* $w \in (0, \infty)$*;* $u'(0) = \infty$*; and* $u(0) = -\infty$*.*

There are two notable parts of this assumption. One is risk aversion, which is critical for the main results. If workers were risk neutral, the optimal wage path would be a step function; i.e., the wage would be zero initially, followed by a jump to a permanent level (see Stevens, 2004). Risk aversion ensures that such jumps are not optimal. Another notable part of the assumption is $u(0) = -\infty$. As discussed extensively by BC, if $u(0) < \infty$, then wages may stay at zero for a finite duration and then increase continuously. Although this possibility does not pose a serious problem to the analysis, it is cumbersome to be included.

There are also a large number of identical firms that can enter the market. Entry is competitive: a firm can post a vacancy at a flow cost $k > 0$. As typically assumed in the literature, a firm has a production technology with constant returns to scale and considers different jobs independently. Normalize the production cost to 0. Recruiting firms announce wage-tenure contracts to compete for workers. A contract offered at time s is a time path of wages, $W(s) = \{w(t)\}_{t=s}^{\infty}$, conditional on the continuation of the worker's employment in the firm. Although a worker can quit at any time, the firm is assumed to commit to the contract. Thus, employment is permanent until the worker either quits the firm or dies. For simplicity, there is no exogenous separation other than death.

Let $V(t, s)$ be the remaining value of the contract to a worker whose tenure in the firm is $(t - s)$. This value is the worker's expected utility derived from the remaining contract from t onward, given the worker's optimal quitting strategy in the future. I will refer to an offer by its value to the worker, $V(s, s)$, because this is all that matters to the worker. All offers are bounded in $\left[\underline{V}, \bar{V}\right]$, where

$$\bar{V} = u(\bar{w})/\sigma, \quad \underline{V} = u(\underline{b})/\sigma.$$

Here, $\bar{w}$ is the highest wage which will be given by Lemma 3.3. The upper bound $\bar{V}$ is the lifetime utility of a worker who is employed at the highest wage permanently. The lower bound $\underline{V}$ is the lifetime utility of a worker who has the lowest unemployment benefit forever and who does not have the opportunity to apply to any job. Because an unemployed worker does have the opportunity of application, all equilibrium offers will be strictly higher than $\underline{V}$. I say that a result holds for all V if it holds for all $V \in [\underline{V}, \bar{V}]$.

Both unemployed and employed workers can search for jobs. At any instant, an unemployed worker receives an opportunity to apply to a job with probability λ_0, and an employed worker receives the opportunity with probability λ_1. I allow for the possibility $\lambda_0 = \lambda_1 = 1$ by letting λ_0, $\lambda_1 \in (0, 1]$.[3] A worker who receives the application opportunity observes all firms' offers instantly without any cost and then chooses the offer to which he applies. As in most search models, each worker can apply to only one offer.[4]

Individuals cannot coordinate their actions. This coordination failure generates unemployment. It is probable that two or more workers apply to the same offer, in which case the firm randomly selects one to employ. If the selected worker is employed elsewhere, the worker must quit that job before accepting the offer. As discussed in the introduction, the worker's incumbent firm is assumed not to respond to the worker's outside offers. A job is destroyed when either the worker accepts another firm's offer or the worker dies.

Because workers observe the offers before they apply, the offers can direct workers' search. Workers and firms both make the tradeoff between an offer and the matching rate at that offer. When choosing a value to offer, a firm faces a *hiring rate* function, $q(.)$. That is, by changing the offer, a firm knows that its hiring rate will change according to $q(.)$. Similarly, an applicant understands that different offers are associated with different employment rates according to an *employment rate* function, $p(.)$. Note that p and q are Poisson rates instead of probabilities, and so they can exceed one.

The functions $q(.)$ and $p(.)$ are equilibrium objects, since they must satisfy two equilibrium requirements. First, they must be consistent with aggregation. That is, as firms and workers make their choices under these functions, the resulting matching rates must indeed be given by these functions. Second, the hiring rate function must ensure that the

[3] Note that the λ'es are not Poisson rates, but rather the probabilities of receiving a job application opportunity at any instant. As such, they are bounded above by one.

[4] Let me clarify two assumptions here. One is that an applicant observes all offers. This assumption is not necessary, because the essential results in directed search are the same if each applicant is assumed to observe two offers that are randomly drawn from the offer distribution (see Acemoglu and Shimer, 1999b). The second assumption is that each applicant can apply to only one offer at a time. (For a directed search model with multiple applications, see Galenianos and Kircher, 2005). In continuous time, this assumption is not as restrictive as it may sound. Although a worker in reality may be able to send out multiple applications, the probability with which two or more of his applications will be received by different firms at the same instant is negligible.

expected profit of recruiting be the same for all equilibrium offers. Delaying the second requirement to section 4, I specify the first requirement below.

Let me start with a matching function, $M(\theta, 1)$, which specifies the measure of matches between a measure θ of workers and a unit measure of firms. Refer to θ as the tightness. Assume that M is linearly homogeneous. Given the two functions $p(.)$ and $q(.)$, individuals' decisions generate the tightness, $\theta(V)$, at each value V. Aggregate consistency requires that the matching rates satisfy: $q(V) = M(\theta(V), 1)$ and $p(V) = M(\theta(V), 1)/\theta(V)$. Using these relationships to eliminate θ, I can express aggregate consistency as $p(V) = P(q(V))$.

The function $P(q)$ embodies all essential properties of the matching function. From now on, I will take $P(q)$ as a primitive of the model and refer to it as the *matching function*.[5] To specify the properties of the matching function, let $q(V) \in [\underline{q}, \bar{q}]$ for all V, with $0 < \underline{q} < \bar{q}$, where $\bar{q}$ is a natural upper bound on q given by the matching function and $\underline{q}$ will be restricted by (5.4) later.

Assumption 2. *The matching function $P(q)$ has the following features: (i) $P(q)$ is continuous for all $q \in [\underline{q}, \bar{q}]$ and, for all q in the interior of $(\underline{q}, \bar{q})$, the derivatives $P'(q)$ and $P''(q)$ exist and are finite; (ii) $\bar{q} < \infty$ and $P(\bar{q}) = 0$; (iii) $P'(q) < 0$; (iv) $-qP''(q)/P'(q) \leq 2$.*

Part (i) is a regularity condition that is satisfied by many well-known matching functions. Part (ii) is imposed for the convenience of working with bounded functions. Part (iii) is equivalent to $0 < \theta M_1/M < 1$, which is satisfied by all matching functions of constant returns to scale that are strictly increasing in the arguments. In equilibrium, I will show $q'(V) > 0$. Then, part (iii) ensures $p'(V) < 0$. Part (iv) restricts convexity of $P(q)$, which will be useful for ensuring uniqueness of a worker's application decision.[6]

Assumption 2 is satisfied by the so-called urn-ball matching function endogenously derived by Peters (1991) and Burdett et al. (2001). To see the assumption in a different example, consider the matching function with a constant elasticity of substitution:

Example 2.1. *If $M(\theta, 1) = [\alpha\theta^\rho + 1 - \alpha]^{1/\rho}$, where $\alpha \in (0, 1)$ and $-\infty < \rho < 1$, then*

$$P(q) = q\left(\frac{q^\rho - 1}{\alpha} + 1\right)^{-1/\rho}$$

[5] I follow the approach in Moen (1997) and Acemoglu and Shimer (1999a) to take the matching function as given. In contrast, some directed search models have gone one step further to derive the matching function endogenously by aggregating agents' strategies, e.g., Peters (1991), Burdett et al. (2001), Julien et al. (2000) and Delacroix and Shi (2006). The main results of the current paper continue to hold when the matching function is endogenized so.

[6] For a general matching function, part (iv) of the assumption requires $1 - \theta M_1/M \leq [-\theta M_{11}/(2M_1)]^{1/2}$, where the left-hand side of the inequality is the share of vacancies in the matching function.

Parts (i) and (iii) of Assumption 2 are satisfied. Part (ii) is satisfied iff $-\infty < \rho < 0$, i.e., iff the elasticity of substitution between searching workers and vacancies is less than one. In this case, $\bar{q} = (1-\alpha)^{1/\rho}$. Part (iv) is satisfied iff $\alpha \geq 1 - (1-\rho)q^{\rho}/2$. When $\rho \leq -1$, this condition is satisfied for all $\alpha > 0$. When $-1 < \rho < 0$, the condition puts a lower bound on α. Note that, for $\rho < 0$, the derivative $P'(q)$ is unbounded at $q = \bar{q}$.

3. Workers' and Firms' Optimal Decisions

In this section, I will characterize agents' optimal decisions and their value functions. Throughout this paper, denote $\dot{x} = dx/dt$ for any variable x.

3.1. Optimal Application

Workers' search is directed by the employment rate function, $p(V)$, which gives the Poisson rate of getting an offer V. As emphasized before, this function is an equilibrium object. Before analyzing workers' search decisions, I describe the properties of this function by the following lemma, which is an implication of Lemma 5.1 later.

Lemma 3.1. *Under Assumption 2, $p(V)$ is bounded, continuous and concave for all V. Moreover, $p(V)$ is differentiable and strictly decreasing for all $V < \bar{V}$, with $p(\bar{V}) = 0$.*

Examine an applicant at time t, who can be either employed or unemployed. Let $V(t)$ be the value for the worker, or the worker's state, generated by the worker's current contract or unemployment benefit. If the worker is employed, this notation suppresses the starting time of the contract. After receiving a job application opportunity, the expected increase in the value for the worker is:

$$D(V(t)) = \max_{f \in [V(t), \bar{V}]} p(f)\,[f - V(t)]\,. \tag{3.1}$$

Denote the solution as $f(t) = F(V(t))$. Then, F is given implicitly as follows:

$$V = F(V) + \frac{p(F(V))}{p'(F(V))}. \tag{3.2}$$

The following lemma, proven in Appendix A, describes the main features of optimal application with directed search. The lemma can be viewed as a generalization of the results from the space of wage levels, such as those in Delacroix and Shi (2006) and Galenianos and Kircher (2005), to the space of values and contracts.

Lemma 3.2. *$F(\bar{V}) = \bar{V}$. For all $V < \bar{V}$, the following results hold: (i) There is a unique and interior solution to (3.1), $f = F(V)$; (ii) $F(.)$ is continuous and $D(V)$ is differentiable, with $D'(V) = -p(F(V)) < 0$; (iii) $F(.)$ is strictly increasing; (iv) $F(V)$ obeys (3.2) and satisfies $[F(V_2) - F(V_1)] / (V_2 - V_1) \leq 1/2$ for all $V_2 \neq V_1$, and so $F(.)$ is a Lipschitz function. If, in addition, $p(.)$ is twice continuously differentiable, then $F(V)$ is differentiable with $0 < F'(V) \leq 1/2$, and $D(V)$ is twice differentiable.*

For a worker at a value V, applying to the offer $F(V)$ is the only optimal choice. This is true despite the fact that the worker observes all other offers. Offers higher than $F(V)$ have too low employment probabilities to be optimal, while offers lower than $F(V)$ have too low values. Only the offer $F(V)$ provides the optimal tradeoff between the value and the probability of obtaining it.

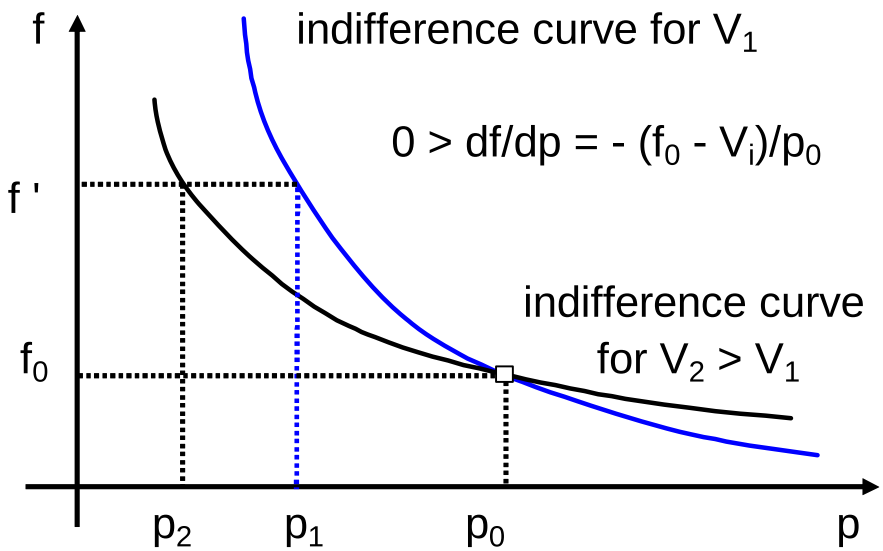

Figure 1. Monotonicity of a worker's optimal application

Not only is a worker's optimal application unique, it is also increasing in the worker's state. That is, the higher the worker's current value, the higher the offer to which the worker will apply. Thus, the workers choose to separate themselves in the application process according to their states. This separation is optimal because an applicant's payoff function has the single-crossing property. The higher a worker's state, the more he can tolerate the risk of not getting an offer, and hence the higher the offer to which he will apply. This difference in the tolerance of risks is not a feature of preferences, but rather a consequence of the lack of insurance. When a worker fails to get the offer to which he applies, he has to fall back onto his current contract or unemployment benefit. The better this backup is, the more a worker can afford to "gamble" on the application, and hence the higher the offer to which he will apply.

Figure 1 illustrates the single-crossing property with two workers, 1 and 2. Worker 1 is at a value V_1 and worker 2 at value V_2, where $V_2 > V_1$. Worker i's indifference curve can be written as $f = V_i + D_i/p$, for $i = 1, 2$. Suppose that the two indifference curves cross each other at a point, (f_0, p_0), where $f_0 > V_2$. At this point, the slope of worker i's indifference curve is $df/dp < 0$, and the absolute value of this slope decreases with V_i. This implies that the worker with the higher value (worker 2) is willing to tolerate a lower employment probability than does the worker with the low value. Equivalently, to compensate for the same reduction in the probability of getting an offer, worker 2 needs a smaller increase in the offer than worker 1 does.

The optimality of the application decision is one of the key differences between this model and the BC model or, more generally, between directed search and undirected search. Models with undirected search have no counterpart to the above decision problem by an applicant. Instead, they assume that each applicant applies to a value which is randomly drawn from the offer distribution. Clearly, such an application strategy is not optimal.

This contrast between the two models leads to sharply different predictions on job-to-job transitions and wage mobility. Directed search predicts a definite pattern of transition and endogenously limited mobility in wages between jobs. For example, take two workers whose current wages are w_1 and w_2, respectively, with $w_1 < w_2$. Let w_A be the starting wage of the contract to which worker 1 chooses to apply, and w_B ($> w_A$) be the starting wage of the contract to which worker 2 chooses to apply. For these two workers, the probability of transiting immediately to a wage above w_B is zero. Moreover, conditional on that both have transited to new jobs, the likelihood between worker 2's and worker 1's probability of having transited to a wage $w' \in (w_A, w_B)$ is zero. In undirected search models, the probability of transiting to wages above w_B is positive for both workers, and the likelihood ratio of transiting to w' is a positive and finite constant.

In addition to limited wage mobility, directed search also yields predictions on the gain to a worker from an application.[7] First, a worker who has a high current value gains less from an application than a worker who has a low current value. This is true in terms of the expected gain from an application, because $D'(V) < 0$. The result is also true in terms of the actual gain in percentage, $(F - V)/V$, because $F'(V) \leq 1/2 < F(V)/V$. With

[7] Delacroix and Shi (2006) establish similar features in a model with directed, on-the-job search, but they restrict that offers must be a constant wage over time. Nevertheless, the similarity suggests that these features are common in directed search models.

risk aversion, however, this decreasing gain in the value does not necessarily translate into a decreasing gain in wages. The decreasing gain in the value partly reflects the worker's decreasing marginal utility as the wage increases. Second, $D''(V) > 0$. That is, the decrease in the expected gain from an application slows down as the worker's current value increases.

3.2. Value Functions of Workers and Firms

For an employed worker, the value can change over time for four possible reasons. The first is the change in wages during the contract with the same firm. The second is the event that the worker obtains a better offer and quits the current job.[8] The third is death. The fourth is the adjustment to the steady state. As in the literature, I abstract from the last source of changes in the value by focusing on a stationary equilibrium. Because the rate of time preference is zero, the value for an employed worker evolves as follows:

$$\dot{V}(t) = \sigma V(t) - u(w(t)) - \lambda_1 D(V(t)) \tag{3.3}$$

If wages were constant over tenure, then $\dot{V} = 0$.

In contrast to wages, the unemployment benefit does not change over time once it is drawn. Thus, the value to an unemployed worker with a given benefit, b, will be constant over time as long as he stays unemployed. Denote this value as $V_u(b)$. Then,

$$0 = \sigma V_u(b) - u(b) - \lambda_0 D(V_u(b)). \tag{3.4}$$

To characterize a firm's value function, consider a firm that is employing a worker at time t under a contract whose remaining value to the worker is $V(t)$. (Again, I suppress the starting time of the contract in this notation.) Let $J(t)$ denote this firm's value. Because the worker quits at rate $\lambda_1 p(F(V(t)))$ and dies at rate σ, then

$$\dot{J}(t) = [\sigma + \lambda_1 p(F(V(t)))]\, J(t) - y + w(t). \tag{3.5}$$

This equation has embodied the aforementioned assumptions that a firm commits to the contract and that it does not respond to the worker's outside offers.

For dynamic optimization, it is useful to express the firm's values as the discounted sum of profits. To do so, let t_0 be an arbitrary point in $[s, t]$, where $s \le t$ is the starting

[8] The worker can also choose to quit the job to become unemployed if the wage profile is sufficiently decreasing. However, this event will never occur in the equilibrium, because the optimal wage profile has increasing wages with tenure, as shown later.

time of the contract. Let $\gamma(t, t_0)$ be the probability that a worker will still be with the firm at time t given that he is with the firm at t_0. Then,

$$\gamma(t, t_0) = e^{-\int_{t_0}^{t} [\sigma + \lambda_1 p(F(V(\tau)))] d\tau}. \tag{3.6}$$

Equivalently, γ is given by the solution to the following differential equation:

$$\frac{d\gamma(t, t_0)}{dt} = -\left[\sigma + \lambda_1 p(F(V(t)))\right] \gamma(t, t_0), \tag{3.7}$$

where $\gamma(t_0, t_0) = 1$ and $\gamma(\infty, t_0) = 0$. Because J is bounded, it satisfies the transversality condition $\lim_{t\to\infty} J(t)\gamma(t, t_0) = 0$. Integrating (3.5) yields:

$$J(t_0) = \int_{t_0}^{\infty} \left[y - w(t)\right] \gamma(t, t_0) dt.$$

For any $t_0 \geq s$, this value is determined by the remaining contract from t_0 onward.

3.3. Optimal Recruiting Decisions and Contracts

Take an arbitrary time $s \geq 0$. A firm's recruiting decision at time s contains two parts. The first part is to choose a value $V(s)$ at which to recruit. The optimal choice maximizes the firm's expected value, $q(V(s))J(s)$, taking the function $q(V)$ as given. As I will explain later, the solution to this part of the firm's problem is a continuum of positive values of $V(s)$. The second part of a firm's problem is to choose a wage profile (i.e., a contract) to maximize $J(s)$ and to deliver the value $V(s)$. I characterize this decision below.

The optimal contract, $\{w(t)\}_{t=s}^{\infty}$, solves:

$$(\mathcal{P}) \ \max \ J(s) \text{ s.t. } (3.3) \text{ for all } t \geq s.$$

In this problem, $V(s)$ is taken as given, and so the maximized value of $J(s)$ is a function of $V(s)$. I express this fact by writing $J(s)$ as $J(V(s))$.

Treat $\gamma(t, s)$ as an auxiliary state variable in the dynamic optimization and (3.7) as the law of motion of γ. Then, the Hamiltonian of the dynamic optimization is:

$$\mathcal{H}(t, s) = (y - w)\gamma(t, s) - \Lambda_\gamma \left[\sigma + \lambda_1 p(F(V))\right] \gamma(t, s) + \Lambda_V \left[\sigma V - u(w) - \lambda_1 D(V)\right],$$

where Λ_γ and Λ_V are shadow prices of γ and V.[9] I suppressed time on the right-hand side, except for γ. Following a similar argument to that in BC, it can be shown that

[9]It can be shown that the program $(\mathcal{P})$ is concave in terms of Gâteaux derivatives in a neighborhood of the optimal contract, and so the optimum is characterized by optimality conditions below.

the assumption $u(0) = -\infty$ implies $w(t) > 0$ for almost all t in all optimal contracts. Optimality conditions are: $\Lambda_\gamma = J$, $\Lambda_V = -\gamma/u'(w)$ and

$$\dot{w} = \frac{[u'(w)]^2}{u''(w)} \lambda_1 J(V) \left[\frac{dp(F(V))}{dV} \right]. \tag{3.8}$$

Optimal contracts have three important properties. First, an optimal contract provides optimal sharing of the value between a firm and its worker. To express this feature formally, note that the Hamiltonian is zero at the optimum.[10] Thus, an optimal contract satisfies:

$$-\dot{J} = \frac{1}{u'(w)} \dot{V}. \tag{3.9}$$

To explain, suppose that the contract increases the value to the worker by a marginal amount, $\dot{V}$. This will entail an increase in the wage by an amount, $\dot{V}/u'(w)$. The cost to the firm, in terms of lost profit, is $-\dot{J}$. The above condition requires that this cost to the firm should be equal to the marginal benefit to the worker.

For the analysis later, it is useful to substitute (3.5) and (3.3) to rewrite (3.9) as:

$$u'(w)(y - w) + u(w) = u'(w) \left[\sigma + \lambda_1 p(F(V)) \right] J(V) + \left[\sigma V - \lambda_1 D(V) \right] \tag{3.10}$$

The best way to explain this equation is to view a match as a joint asset. With this view, the left-hand side of the equation measures the flow of "dividends" to the asset, which consists of the firm's profit, evaluated with the worker's marginal utility, and the worker's utility of the wage. The right-hand side is the "permanent income" in utils generated by the asset. In particular, the permanent income to the firm is $[\sigma + \lambda_1 p(F)]J$, which is translated into units of utility with the marginal utility of the worker. The permanent income to the worker is $[\sigma V - \lambda_1 D(V)]$. The optimal contract requires that the flow of dividends to the joint asset should be equal to the permanent income of the asset.

Second, an optimal contract provides wages that increase with tenure. This feature and the bounds on wages are stated as follows (see Appendix B for a proof):

Lemma 3.3. *$\dot{w}(t) > 0$ for all $V < \bar{V}$. Moreover, $\bar{w} = y - \sigma k/\bar{q} < y$, $\bar{V} = u(\bar{w})/\sigma$, $J(\bar{V}) = k/\bar{q} > 0$, and $q(\bar{V}) = \bar{q} < \infty$.*

There are two forces that make an optimal wage profile increase smoothly with tenure. The first is a firm's incentive to retain a worker in the absence of commitment by the

[10] To obtain this result, differentiate the Hamiltonian with respect to time, and then substitute (3.3), (3.5) and the optimality conditions. This shows that the Hamiltonian, $\mathcal{H}(t, s)$, is constant over t. Because $\gamma(\infty, s) = 0$, then $\mathcal{H}(t, s) = \mathcal{H}(\infty, s) = 0$ for all $t \geq s$.

worker, and the second is a worker's risk aversion. Because a worker cannot commit to the job, a firm can increase the worker's opportunity cost of quitting by backloading wages. As wages rise with tenure, the probability with which the employee can find a better offer elsewhere falls, and so the worker's quit rate falls with tenure. Thus, a rising wage profile is less costly to the firm than a constant wage profile that provides the same expected value to the worker. However, if workers are risk neutral, then the best way for a firm to backload wages is to offer zero wage initially with a promised jump in wages in the future (see Stevens, 2004). This jump is not desirable for risk averse workers and so, for such workers, the optimal contract has smoothly increasing wages over tenure. These two forces appear in (3.8): the incentive to retain a worker appears through the derivative $dp(F(V))/dV$ (< 0) and risk aversion through $u'' < 0$.

Because wages are increasing with tenure and bounded above, wages in all optimal contracts increase toward the upper bound $\bar{w}$ as $t \to \infty$. Accordingly, the value for an employed worker converges to $\bar{V}$.[11] This convergence in the value is also monotonic, as I will show later in Corollary 5.3. As a result, a firm's value falls over time.

The third property is that all optimal contracts are sections of a baseline contract. To describe this property, let the baseline contract be $\{w_b(t)\}_{t=0}^{\infty}$, where $w_b(0)$ is the lowest wage in equilibrium. Every other optimal contract, $\{w(t)\}_{t=0}^{\infty}$, traces out the baseline contract from a particular initial wage. That is, the entire set of optimal contracts is:

$$\left\{\{w(t)\}_{t=0}^{\infty} : w(t) = w_b(t+s) \text{ for all } t, \text{ where } s \in [0, \infty)\right\}.$$

This property is an implication of the principle of dynamic optimality. To explain why, note that, once V is given, the firm's optimization problem does not depend on the starting time of the contract, s. Consider two contracts: contract 1 is offered at time s_1 and contract 2 offered at $s_2 > s_1$. The value offered by contract 2 is V_2. Suppose that contract 1 from s_2 onward also delivers V_2, then the remaining part of contract 1 must be the same as contract 2. Otherwise, the firm that offers contract 1 could replace the remaining part of the contract by contract 2 which, by the optimality of contract 2 from s_2 onward, would improve the firm's expected value.

This property of dynamic optimality simplifies the analysis greatly. One simplification is that characterizing the entire set of optimal contracts at any time is equivalent to tracing out the baseline contract over time. Similarly, characterizing the set of offer values at any time is equivalent to tracing out the values provided by the baseline contract over time.

[11] Offers above $\bar{V}$ are not optimal because they generate expected values to the firm that are less than the recruiting cost.

From now on, I will focus on the baseline contract, suppress the subscript b, and suppress the starting point of a contract.

Another simplification is that the wage at any tenure can be written as a function of the value remaining in the contract, rather than a function of tenure. To do so, let $\mathcal{V}$ be the set of equilibrium lifetime utilities. Define $v_1 = \inf(\mathcal{V})$ and define T by

$$T(V(t)) = t, \text{ with } T(v_1) = 0. \tag{3.11}$$

Then, $T(x)$ is the length of tenure required for a worker to increase the value from v_1 to x according to the baseline wage contract. The wage level of a worker with tenure t on the baseline contract is $w(T(V(t)))$. With a slight abuse of the notation, I express this wage as $w(V)$ and refer to the function as the *wage function*. The above explanation makes it clear that $w(V(t))$ is also the starting wage of a contract that is offered at t with a value $V(t)$ to the worker. The notation $w(V)$ should be construed to mean that wage can only vary over time when the value to the worker changes over time.

Similarly, the notation $J(V)$ indicates that a firm's value can only change over time when the value to the worker changes over time. Thus, I can rewrite (3.9) as

$$J'(V) = -\frac{1}{u'(w(V))} < 0. \tag{3.12}$$

4. Definition and Configuration of the Equilibrium

Let n be the fraction of workers who are employed and $(1-n)$ the fraction of workers who are unemployed. Let G_e be the cumulative distribution function of employed workers over values and G_u be the distribution of unemployed workers over values.

An equilibrium is a set of lifetime utilities, $\mathcal{V}$, a Poisson rate of employment, $p(.)$, an application strategy, $F(.)$, a value function $J(.)$, a wage function $w(.)$, and distributions of workers, (G_e, G_u, n), that satisfy the following requirements:

(i) G_e, G_u and n are stationary;

(ii) $F(V)$ solves (3.1), given $p(.)$;

(iii) Given $F(.)$ and $p(.)$, each value $V \in \mathcal{V}$ is delivered by a contract that solves $(\mathcal{P})$ for $s = 0$ with a starting wage $w(V)$, and the resulting value function of the firm is $J(V)$;

(iv) Zero expected profit of recruiting: $q(V)J(V) = k$ for all $V \in [\underline{V}, \bar{V}]$, and $q(V)J(V) < k$ for all $V > \bar{V}$, where $q(V) = P^{-1}(p(V))$.

Most elements of this definition are self-explanatory, but requirement (iv) needs clarification. This requirement asks the function $q(V)$ to induce zero expected profit from recruiting for all $V \in [\underline{V}, \bar{V}]$, not just for $V \in \mathcal{V}$. Since $\mathcal{V}$ is a strict subset of $[\underline{V}, \bar{V}]$, as I will argue after (6.1), the requirement imposes a restriction on beliefs out of the equilibrium. The reason for imposing this restriction is as follows. For a non-equilibrium value $V \notin \mathcal{V}$, there can be two different reasons why the value is not in the equilibrium set. One is the self-fulfilling expectation that no worker will apply to that value: This expectation induces firms not to offer that value, in which case no worker will apply to that value, indeed. The second reason is that, even after firms offer that value, workers still find it optimal not to apply to it. The first reason for a "missing market" may not be robust to a trembling event that exogenously puts some recruiting firms at the value V. Requirement (iv) excludes such non-robust equilibria, and hence, refines the set of equilibria. This refinement resembles trembling-hand perfection.[12]

Requirement (iv) determines the hiring rate function, and hence, the employment rate function. For given $J(.)$, the requirement yields $q(V) = k/J(V)$, and so $p(V) = P(k/J(V))$ for all $V \in [\underline{V}, \bar{V}]$. For all $V > \bar{V}$, (iv) requires that a firm recruiting at V should make an expected loss. This part of the requirement is always satisfied, because Lemma 3.3 implies $q(V)J(V) \leq \bar{q}J(V) < \bar{q}J(\bar{V}) = k$.

I illustrate the configuration of the equilibrium in Figure 2. The set of equilibrium values for employed workers is $\mathcal{V} = [v_1, \bar{V}]$ and the set of equilibrium values for unemployed workers is $[v_0, \bar{V}]$, where $v_1 = F(v_0) > v_0 > \underline{V}$ and v_0 will be defined later by (6.1). The arrows in Figure 2 depict the applications of the workers at the special values v_j, where $v_{j+1} = F(v_j)$ with $j = 0, 1, 2, ...$ A worker at v_j applies to v_{j+1}. If he gets the job, his value jumps to v_{j+1}. If he does not get the job, his value increases smoothly above v_j according to the contract and the target of his application increases above $F(v_j)$. Thus, the gap between any two special values, v_j and v_{j+1} (with $j \geq 1$), is filled in by workers who have stayed with their contracts for some time. There are employed workers at *every level* in $[v_1, \bar{V}]$ and unemployed workers at every level in $[v_0, \bar{V}]$. A worker whose value lies in the interior of (v_j, v_{j+1}) chooses to apply to a unique value in the interior of (v_{j+1}, v_{j+2}).

[12]Similar refinements have been used in directed search models, e.g., Acemoglu and Shimer (1999b) and Delacroix and Shi (2006). In Delacroix and Shi, the refinement restricts the applicants', rather than firms', expected payoff off the equilibrium path. It requires a worker's expected surplus from applying to every offer (including a non-equilibrium offer) to be the same. In an environment with homogeneous workers, this alternative restriction achieves the same purpose as requirement (iv) does. However, when the applicants are heterogeneous as in the current model, the alternative restriction is not useful because it is not possible to find one function $p(.)$ or $q(.)$ that induces all applicants to be indifferent between equilibrium and non-equilibrium offers.

This application is not depicted in Figure 2.

As emphasized earlier, the unique choice of each worker's application contrasts with the application strategy in undirected search models, such as Burdett and Mortensen (1998) and BC, where a worker sends the application to a randomly selected value in $[v_1, \bar{V}]$. This difference in the nature of search implies that the two types of models work in very different ways. For an equilibrium with undirected search, the most important objects are the distributions of values and workers, which affect the employment rate and the hiring rate. In contrast, for an equilibrium with directed search, the most important objects are the employment rate function, $p(V)$, and the hiring rate function, $q(V)$. One can determine these functions, and hence determine optimal application and hiring decisions, by invoking requirements (ii) – (iv) of the above definition, *without* any explicit reference to the distributions of offers and workers.

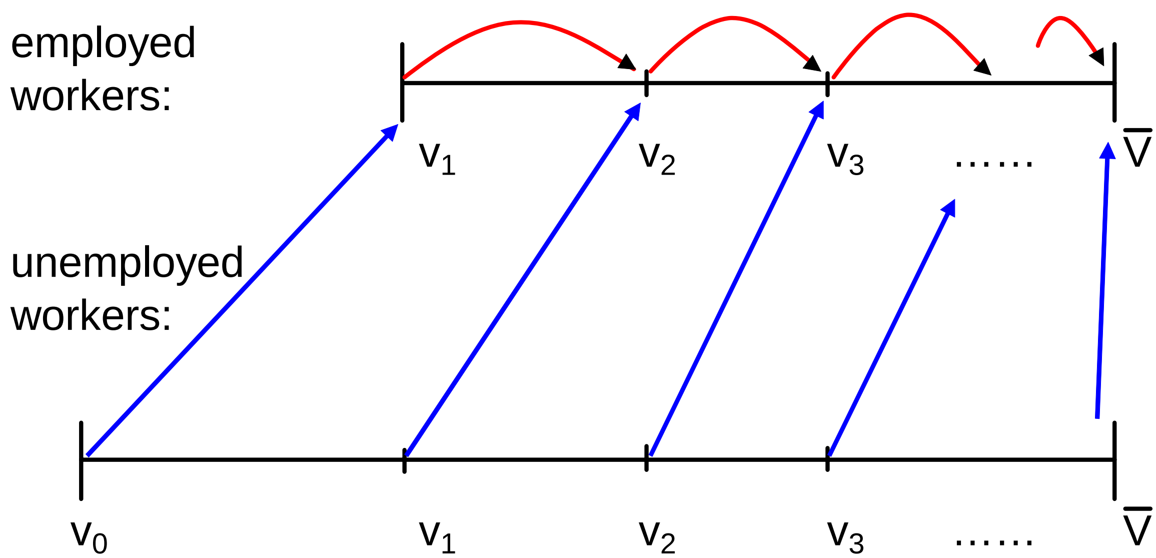

Figure 2. An illustration of the equilibrium

Now, it is useful to clarify the role of the assumption of a continuous distribution of unemployment benefits. The main use of this assumption is to make the distribution of employed workers smooth (i.e., differentiable) at the critical levels v_j for $j \geq 3$. As is clear from the above discussion of the equilibrium definition, the assumption does not play any role in the characterization of the wage function, contracts, or individuals' matching rates in the equilibrium. Moreover, the assumption is not necessary for existence of a continuous wage distribution. Even if all unemployed workers have the same benefit, wage-tenure contracts and search on the job will continue to generate a continuous wage distribution in equilibrium. As in Delacroix and Shi (2006), search on the job enables workers who luckily got jobs earlier to apply to higher wages than those who get jobs later. Furthermore (and in contrast with Delacroix and Shi), because wages increase with tenure, those workers who got jobs earlier have had their wages grow to higher levels than those who got jobs later. With continuous time, these features generate a continuous wage distribution.

5. Equilibrium Employment Rate and the Wage Function

The main step of determining an equilibrium is to determine the employment rate function, $p(V)$. However, it is more convenient to build the proof of existence around the wage function. The following procedure develops a mapping for w and obtains other functions, (p, J, F). The procedure will also be useful for comparative statics later.

Start with any function $w(.)$ and add the subscript w to other functions constructed with this given function. First, given $w(.)$, I integrate (3.12) and use $J(\bar{V}) = k/\bar{q}$ to get:

$$J_w(V) = k/\bar{q} + \int_V^{\bar{V}} \frac{1}{u'(w(z))} dz \tag{5.1}$$

Second, the zero-profit condition for recruiting yields $q_w(V) = k/J_w(V)$ and hence

$$p_w(V) = P\left(\frac{k}{J_w(V)}\right) \tag{5.2}$$

Third, using $p_w(V)$ as the employment rate, I can express an applicant's optimal decision as $f = F_w(V)$ and the expected gain as $D_w(V)$. Fourth, I explore (3.10), a requirement on a firm's optimal recruiting decision. Treat w on the left-hand side of (3.10) as a variable but substitute the given function $w(V)$ for w on the right-hand side. To avoid confusion, use w_1 instead of w on the left-hand side. Then,

$$u(w_1) + u'(w_1)(y - w_1) = u'(w(V)) \left[\sigma + \lambda_1 p_w(F_w(V))\right] J_w(V) + \sigma V - \lambda_1 D_w(V) \tag{5.3}$$

Denote the solution for w_1 as $w_1(V) = (\Gamma w)(V)$. Equilibrium wage function, $w(V)$, is a fixed point of the mapping Γ. That is, $w(V) = (\Gamma w)(V)$ for all V.

Confirming an earlier statement, the above procedure does not involve the distributions of workers and offers. Thus, optimal contracts and applications are independent of such distributions. I will explore this feature of the equilibrium later in section 7.

To characterize the fixed point for w, let me specify the bounds on various functions. First, using the constant $\bar{w}$ to replace the function $w(V)$ in (5.1) and (5.2), I obtain $J_{\bar{w}}(V)$ and $p_{\bar{w}}(V)$. Because $J_w(.)$ and $p_w(.)$ are monotone in w, then $J_w(V) \leq J_{\bar{w}}(V)$ and $p_w(V) \leq p_{\bar{w}}(V)$ for all V. Second, define

$$\underline{q} = k/J_{\bar{w}}(\underline{V}). \tag{5.4}$$

Since $J_{\bar{w}}(V)$ is decreasing, $q(V) \in [\underline{q}, \bar{q}]$ for all V, and $\underline{q} \in (0, \bar{q})$. This lower bound on q is the one used in Assumption 2. Similarly, $p(V)$ is bounded in $[0, P(\underline{q})]$. Third, let $\underline{w}$ be a strictly positive number that is sufficiently close to 0.

Assumption 3. *Assume that $\underline{b}$, $\underline{V}$ and $\underline{w}$ satisfy:*

$$(0 <)\ \underline{b} < \bar{w} = y - \sigma k/\bar{q} \tag{5.5}$$

$$J_{\bar{w}}(\underline{V})\left[\sigma + \lambda_1 p_{\bar{w}}(\underline{V})\right] < y \tag{5.6}$$

$$u(\underline{w}) + u'(\underline{w})\left[y - \underline{w} - J_{\bar{w}}(\underline{V})(\sigma + \lambda_1 p_{\bar{w}}(\underline{V}))\right] \geq u(\underline{b}). \tag{5.7}$$

The condition (5.5) is a regularity condition: When it is violated, all workers will choose to stay out of employment. The condition (5.6) requires that the permanent income of a job to a firm be less than output even when the firm is providing the lowest value to the worker. To see which parameters this condition restricts, note that $J_{\bar{w}}(V)$ and $p_{\bar{w}}(V)$ are decreasing functions. Then, the left-hand side of (5.6) is decreasing in $\underline{V}$, and hence decreasing in $\underline{b}$. As a result, (5.6) is satisfied if $\underline{b}$ is bounded below by some number. If I set $\underline{b} = \bar{w}$, the left-hand side of (5.6) is equal to $\sigma k/\bar{q}$, which is less than y by (5.5). Thus, there exists $\hat{b} \in (0, \bar{w})$ such that (5.5) and (5.6) are satisfied if $\underline{b} \in (\hat{b}, \bar{w})$.

To see what (5.7) entails, note that the left-hand side of (5.7) is a decreasing function of $\underline{w}$ for sufficiently small $\underline{w}$. Thus, (5.7) imposes an upper bound on $\underline{w}$. Because $\underline{w}$ is chosen to be sufficiently close to 0, a sufficient condition for (5.7) is:

$$\lim_{w \downarrow 0} \left[u(w) + u'(w)(a - w)\right] = \infty \text{ for all } a > 0.$$

This sufficient condition is satisfied by the example $u(w) = \left(w^{1-\eta} - 1\right)/(1-\eta)$ with $\eta > 1$.

Define

$$\begin{aligned} \Omega = \ & \{w(V) : w(V) \text{ is continuous and (weakly) increasing;} \\ & \qquad w(V) \in [\underline{w}, \bar{w}] \text{ for all } V;\ w(\bar{V}) = \bar{w}\Big\} \\ \Omega' = & \Big\{w \in \Omega : w(V) \text{ is strictly increasing for all } V < \bar{V}\Big\}. \end{aligned}$$

I establish that a fixed point of Γ exists in Ω and then show that it lies in the subset Ω'. First, the following lemma holds (see Appendix B for a proof):

Lemma 5.1. *For any $w \in \Omega$, $J_w(V)$ defined by (5.1) is strictly positive, bounded, strictly decreasing and continuously differentiable for all V. The function $p_w(V)$ defined by (5.2) has all the properties stated in Lemma 3.1.*

This lemma shows that $p_w(V)$ has all the properties that enable parts (i) - (iii) in Lemma 3.2 to hold. As a result, there is a unique and interior solution to (3.1), $F_w(V)$, which is continuous and strictly increasing for all $V < \bar{V}$. Moreover, $D'_w(V) = -p_w(F_w(V)) < 0$.

Theorem 5.2. *Maintain Assumptions 1, 2 and 3. Assume that the image of* Γ *is compact. Then, the mapping* Γ *has a fixed point,* $w^* \in \Omega'$*. That is,* $w^*(V)$ *is continuous on* $[\underline{V}, \bar{V}]$*, its values lie in* $[\underline{w}, \bar{w}]$ *with* $w^*(\bar{V}) = \bar{w}$*, and it is strictly increasing for all* $V < \bar{V}$*. The implied functions* $J_{w^*}(V)$ *and* $p_{w^*}(V)$ *are strictly concave, in addition to the properties stated in Lemma 5.1.*

Proof. See Appendix C.

In the remainder of this paper, I will suppress the * on the fixed point and the subscript w^* on the equilibrium functions J, p, F and D.

The above theorem establishes continuity, but not differentiability, of the wage function. However, differentiability is useful for various parts of the analysis. Moreover, I need to confirm that the value to a worker, as well as the wage, increases with tenure. To deliver these features, I focus on wage profiles that are smooth over tenure, as the following corollary states (see Appendix D for a proof):

Corollary 5.3. *If* $|\dot{w}(t)| < \infty$ *for all* t*, then* $w(V)$ *is differentiable, with* $0 < w'(V) < \infty$ *for all* V*. Moreover, the following results hold for all* $V < \bar{V}$*: (i) the derivatives* $J''(V)$*,* $p''(V)$ *and* $F'(V)$ *exist and are finite; (ii)* $\dot{V} > 0$ *and* $\dot{J}(V) < 0$*.*

It is informative to compare wages with unemployment benefits. To do so, let $B(V)$ be the unemployment benefit starting at which a worker can achieve the lifetime value V. I refer to this function as the *benefit function* and compare it with the wage function. To compute $B(V)$, use (3.4) to solve $V_u = V_u(b)$. Inverting this solution yields:

$$B(V) = u^{-1}\left(\sigma V - \lambda_0 D(V)\right). \tag{5.8}$$

Since $D'(V) < 0$, then $B'(V) > 0$ and $V'_u(b) > 0$. Also, the assumption $\bar{b} = \bar{w}$ implies $V_u(\bar{b}) = \bar{V}$.

For all $V \in [v_1, \bar{V}]$, $\dot{V} \geq 0$, and so the following holds for all $\lambda_0 \leq \lambda_1$:

$$u(w(V)) = \sigma V - \lambda_1 D(V) - \dot{V} \leq \sigma V - \lambda_0 D(V) = u(B(V)).$$

Thus, $w(V) \leq B(V)$. The inequality is strict when $\dot{V} > 0$, i.e., when $V < \bar{V}$. Thus, I have established the following result:

Corollary 5.4. *$w(\bar{V}) = B(\bar{V})$. If $\lambda_0 \le \lambda_1$, then $w(V) < B(V)$ for all $V \in [v_1, \bar{V})$.*

The novel part of this corollary is the case $\lambda_0 = \lambda_1$. In this case, an unemployed worker has the same access to jobs as an employed worker. Yet, the unemployment benefit must be higher than the wage in order for an unemployed worker to achieve the same value V as an employed worker. Put differently, if the unemployment benefit is the same as (or lower than) an employed worker's wage, the present value for the unemployed worker is lower than that for the employed worker. The reason is that an employed worker enjoys the prospect of rising wages while an unemployed worker's benefit does not change over time. This disadvantage of an unemployed worker must be compensated by a higher unemployment benefit in order for the unemployed worker to achieve the same value as an employed worker. This result may hold even for some $\lambda_0 > \lambda_1$.

Of course, if $\lambda_0 < \lambda_1$, then an unemployed worker has a more difficult access to jobs than an employed worker. In this case, there is an additional reason for $B(V) > w(V)$, as in Burdett and Mortensen (1998).

6. Equilibrium Distributions of Workers and Firms

Having determined individuals' optimal decisions and the functions of matching rates, I now compute the distribution of unemployed workers, G_u, and the distribution of employed workers, G_e. Let g_u be the density function corresponding to G_u and g_e corresponding to G_e. Note that these distributions are over values. The distributions over wages or unemployment benefits can be recovered with $w(V)$ and $B(V)$. For example, the distribution of employed wages, denoted as $G_w(w)$, is given by $G_w(w(V)) = G_e(V)$, and the density function is $g_w(w) = g_e(V)/w'(V)$. I do not characterize the distribution of offers, because it is not important for the analysis.

Let me start by defining the following particular values:

$$v_0 = V_u(\underline{b}) \text{ and } v_j = F^{(j)}(v_0), \quad j = 1, 2, ... \tag{6.1}$$

where $F^{(0)}(v_0) = v_0$ and $F^{(j)}(v_0) = F(F^{(j-1)}(v_0))$. The support of G_u is $[v_0, \bar{V}]$ and the support of G_e is $[v_1, \bar{V}]$, as depicted earlier in Figure 2. Clearly, $v_1 > v_0 = \underline{V} + \lambda_0 D(v_0)/\sigma > \underline{V}$. Thus, the set of equilibrium values is a strict subset of $[\underline{V}, \bar{V}]$.

Next, define $\Phi(V) = H(B(V))$ and $\phi(V) = \Phi'(V)$. The function Φ transforms the distribution of unemployment benefits into a distribution of values. Drawing a benefit b according to H is equivalent to drawing a value V according to Φ. Because $B'(V) > 0$, then

$$\phi(V) = h(B(V))B'(V) = \frac{\sigma + \lambda_0 p(F(V))}{u'(B(V))} h(B(V)). \tag{6.2}$$

Because $h(.)$, $F(.)$ and $B(.)$ are differentiable, $\phi(.)$ is differentiable. In Appendix D, I prove the following lemma:

Lemma 6.1. *The distribution of unemployed workers and the fraction of employment are:*

$$G_u(V) = \frac{\sigma}{1-n} \int_{v_0}^{V} \frac{\phi(z)}{\sigma + \lambda_0 p(F(z))} dz. \tag{6.3}$$

$$n = 1 - \sigma \int_{v_0}^{\bar{V}} \frac{\phi(z)}{\sigma + \lambda_0 p(F(z))} dz. \tag{6.4}$$

Now I compute G_e. Examine the group of employed workers whose values are greater than V, where $V \in [v_1, \bar{V}]$. Since death is the only flow out of this group, the outflow from this group in a small interval dt is $\sigma n[1 - G_e(V)](dt)$. There are three flows into the group. One is the group of workers who were employed at or below V and whose values increased above V according the contract. The size of this flow is $n[G_e(V) - G_e(V - \dot{V}dt)]$. The second inflow is the group of workers who were employed at or below V and who received offers above V. This inflow exists only if the workers' values before the application are equal to or greater than v_1, i.e., if $F^{-1}(V) \geq v_1$; otherwise, the workers were unemployed. The third inflow is the group of unemployed workers who received offers above V. Before receiving offers, these workers had values in $[F^{-1}(V), \bar{V}]$. Equating the outflows to the sum of inflows, and taking the limit $dt \to 0$, I get:

$$\begin{aligned} & \sigma n \left[1 - G_e(V)\right] \\ = \; & n \lim_{dt \downarrow 0} \tfrac{G_e(V) - G_e(V - \dot{V}dt)}{dt} + \lambda_1 n \textstyle\int_{\max\{v_1, F^{-1}(V)\}}^{V} p(F(z)) dG_e(z) \\ & + \lambda_0 (1-n) \textstyle\int_{F^{-1}(V)}^{\bar{V}} p(F(z)) dG_u(z). \end{aligned} \tag{6.5}$$

To solve for g_e, partition the support of G_e into subintervals $[v_j, v_{j+1})$, where v_j is defined by (6.1). Add a subscript j to $g_e(V)$ and $G_e(V)$ for $V \in [v_j, v_{j+1})$. Also, define $\Delta(V) = \left[\Phi(V) - (1-n)G_u(V)\right]/n$ and

$$\delta(V) \equiv \Delta'(V) = \frac{\lambda_0 p(F(V)) \phi(V)}{n\left[\sigma + \lambda_0 p(F(V))\right]}. \tag{6.6}$$

I prove the following theorem (see Appendix E for a proof):

Theorem 6.2. *The distribution of employed workers, G_e, does not have a mass point. The density function, $g_e(V)$, is continuously differentiable for all $V \neq v_2$. Moreover,*

$$g_e(V)\dot{V} = \sigma\Delta(F^{-1}(V)) - \sigma G_e(V) - \lambda_1 \int_{\max\{v_1, F^{-1}(V)\}}^{V} p(F(z))dG_e(z). \tag{6.7}$$

With $T(V)$ defined by (3.11), g_e can be solved piece-wise as follows:

$$g_{e1}(V)\dot{V} = \sigma \int_{v_1}^{V} \gamma(T(V), T(z))\delta(F^{-1}(z))dF^{-1}(z), \tag{6.8}$$

$$\begin{aligned} & g_{ej}(V)\dot{V} - g_{ej}(v_j)\dot{v}_j\gamma(T(V), T(v_j)) \\ = \; & \textstyle\int_{v_j}^{V} \gamma(T(V), T(z)) \left\{\sigma\delta(F^{-1}(z)) + \lambda_1 p(z) g_{e(j-1)}(F^{-1}(z))\right\} dF^{-1}(z) \end{aligned} \tag{6.9}$$

where (6.9) holds for $j \geq 2$. Moreover, $g_{ej}(v_j) = \lim_{V\uparrow v_j} g_{e(j-1)}(V)$ for all j.

The theorem gives the following procedure to compute g_e. Starting with $j = 1$, (6.8) gives g_{e1}. Taking the limit $V \uparrow v_2$ in the formula yields $g_{e2}(v_2)$. Then, setting $j = 2$ in (6.9) yields $g_{e2}(V)$. Taking the limit $V \uparrow v_3$ in the result yields $g_{e3}(v_3)$. Continue this process until g_{ej} is obtained for all j.

The following corollary describes the shape of g_e (see Appendix E for a proof):

Corollary 6.3. *$g_e(v_1) = 0$ and $g_e'(v_1) > 0$. If $F'(\bar{V}) > 0$, then $g_e(\bar{V}) = 0$. In this case, there exists $\hat{V} \in (v_1, \bar{V})$ such that $g_e'(V) < 0$ for all $V \in [\hat{V}, \bar{V})$.*

The corollary says that the density function of employed workers is increasing at low values and decreasing at high values. Thus, the density function is non-monotonic, with more workers being employed at intermediate values than at values at the two ends. Note that the density of employed wages is also non-monotonic. To see this, recall that the density of employed wages is $g_w(w) = g_e(V)/w'(V)$. Because $0 < w'(V) < \infty$ by Lemma 5.3, the above corollary yields $g_w(w_1) = g_w(\bar{w}) = 0$, where $w_1 = w(v_1)$. Thus, the shape of $g_w(w)$ at the two ends is similar to the shape of $g_e(V)$ at the two ends. That is, the density of employed wages is increasing when wage is low and decreasing when wage is high.

Before discussing this result, let me check how easily the condition $F'(\bar{V}) > 0$ can be satisfied. Consider the matching function in Example 2.1. Write the first-order condition of a worker's application as $F^{-1}(V) = V + p(V)/p'(V)$. Differentiating this condition and evaluating at $\bar{V}$ yields $dF^{-1}(V)/dV|_{V=\bar{V}} = 1 - \rho$. Thus, with the CES matching function, $F'(\bar{V}) > 0$ is always satisfied under Assumption 2.

The non-monotonicity described in the above corollary is a robust feature of the data. In particular, the wage density is decreasing at high wages (see Kiefer and Neumann, 1993). Directed search is able to capture this feature of the data because workers choose their applications optimally. To see why, consider a worker with a value V and assume $F(V) < \bar{V}$. This worker also observes other offers including those higher than the target value, $F(V)$. The choice of not applying to higher offers is optimal only if higher offers are more difficult to be obtained than the target value. For this to be true, the measure of recruiting firms per applicant must be smaller at high values than at the target value. In particular, at values close to the upper bound $\bar{V}$, the measure of recruiting firms per applicant should be close to zero. In turn, as few workers apply to such high values, it is indeed optimal for only few firms to recruit at these values. The measure of workers who succeed in obtaining jobs at values near $\bar{V}$ is close to zero. This feature makes the density function of employed values decreasing near the upper end of the distribution.

In contrast, undirected search models with homogeneous matches are unable to produce a density function that decreases at high values (or wages). The density function in such models is increasing and convex (see Burdett and Mortensen, 1998, and BC). The cause for this failure is the assumption of undirected search. Under this assumption, all applicants send their applications randomly and uniformly to the recruiting firms. In this case, firms cannot hope to attract more applications by increasing offers. Instead, they use high offer to increase acceptance and retention. For these purposes, a high value is superior to a low value, and so more firms recruit at high values than at low values. The increasing density of recruiting firms ensures that recruiting yields the same expected profit at all equilibrium values. It also results in more workers being employed at high values than at low values, i.e., an increasing density of employed workers. To modify this unrealistic prediction, models of undirected search have introduced heterogeneity across matches in workers' or firms' characteristics (e.g., van den Berg and Ridder, 1998).

The above difference between the two classes of models can be illustrated with the roles of the functions of $p(V)$ and $q(V)$. A directed search model and an undirected search model both require zero net expected profit from recruiting at all values, i.e., $q(V) = k/J(V)$. Both models produce a decreasing and concave value function of firms, $J(V)$, which implies that $q(V)$ is increasing and convex. However, the difference between the two models arises in the link between $q(V)$ and the distribution of workers. This link is tight when search is undirected. In that case, a firm's hiring rate at V is equal to the rate at which the application comes from a worker employed below V. That is,

$$\lambda_1 n G_e(V) + \lambda_0 (1-n) G_u(V) = q(V)$$

Because $q(V)$ is convex, then the density functions $g_e(V)$ and $g_u(V)$ are likely to be both increasing. Directed search breaks this close link between q and G_e. With directed search, the critical determinant of the equilibrium distribution of workers is not the hiring rate, but rather workers' application decisions which is governed by the function $p(V) = P(q(V))$. Although $q(V)$ is still convex, the function $p(V)$ is decreasing and concave. In particular, the employment rate at values close to $\bar{V}$ is almost zero. As a result, few workers are employed at such high values, and so the density of employed workers is decreasing at this high end of the distribution.

Note that Corollary 6.3 holds independently of the shape of the density function of unemployment benefits, $h(b)$. Thus, g_e is non-monotonic here regardless of whether $h(b)$ is increasing, decreasing, flat, or non-monotonic. Heterogeneity in unemployment benefits is not the cause for the feature that the density of employed workers is decreasing at high values (or wages).[13]

7. Comparative Statics

In this section, I conduct two comparative statics, one with respect to the distribution of unemployment benefits and the other with respect to the parameter λ_0. These comparative statics further illustrate the differences between the current model of directed search and undirected search models.

Suppose that the distribution of unemployment benefits, H, increases in the first-order stochastic dominance, with the support remaining unchanged. The effects of this change are summarized in the following corollary:

Corollary 7.1. *An increase of the first-order stochastic dominance in unemployment benefits has no effect on workers' optimal applications and equilibrium contracts. It does not affect the supports of the distributions $G_u(.)$ and $G_e(.)$, either, although it affects the shape of these distribution functions.*

Proof. The analysis in section 5 are independent of the distributions, G_e, G_u and H. Thus, the functions, $w(.)$, $p(.)$, $q(.)$, $J(.)$, $F(.)$, and $D(.)$ after the change in H are all the same as before. Because the function $D(.)$ is independent of H, the function $V_u(.)$ is independent of H. Under the assumption that $\underline{b}$ does not change, v_0 and v_1 do not change, either, because $v_0 = V_u(\underline{b})$ and $v_1 = F(v_0)$. Taken together, these results imply that the

[13]If the distribution of b is degenerate at a particular value, then g_e may be positive at v_1, but g_e will still be decreasing at values close to $\bar{V}$.

change in H has no effect on a worker's optimal application strategy or the equilibrium set of contracts. However, G_u and G_e change with H, because Φ does. QED

The distributions of workers, G_e, G_u and H, do not play any role in the determination of optimal contracts and optimal applications. Although this feature arises from the procedure in section 5, it is worthwhile explaining the feature. To do so, start with a worker's application. For a worker to choose the optimal application, $F(V)$, he only needs the employment rate function. In turn, the employment rate must satisfy the requirement that recruiting should yield zero expected profit at all equilibrium offers. This requirement pins down $p(V)$, given a firm's value function, $J(V)$. However, a firm value function depends only on what happens after the hiring, that is, on the contract offered, $w(V)$, and the worker's quit rate, $p(F(V))$. By the proceeding argument, the two functions that determine the quite rate, $p(.)$ and $F(.)$, are only functions of optimal contracts. Thus, the only thing still to be determined is the set of optimal contracts, i.e., the function $w(V)$. The function $w(V)$ provides efficient sharing of the value between a firm and a worker, in the sense that $-\dot{J} = \dot{V}/u'(w)$. Again, $\dot{J}$ and $\dot{V}$ involve only the functions $F(V)$, $p(V)$, $J(V)$ and $w(V)$ (see (3.3) and (3.5)). The solution to this fixed-point problem is independent of the distributions of employed and unemployed workers.[14]

A strong (testable) implication of the above corollary is that changing unemployment benefits can change wage distributions and the average duration of unemployment, but it does not change individual workers' job-to-job transition rates or their wage paths. Of course, aggregate flows between jobs do change, because they depend on the distribution of workers over the values.

The above predictions are markedly different from those in undirected search models.[15] There, an increase in unemployment benefits reduces the probability with which a given offer will be accepted by a worker, thereby inducing the equilibrium distribution of offers to increase. As more firms are offering higher values than before, the transition rate from low-value jobs to high-value jobs increases. That is, the quitting rate at low-value jobs increases. In order to mitigate this increase in the quitting rate, firms offer contracts in which wages

[14]There are two qualifications. First, the distribution H can affect equilibrium contracts if the number of firms is fixed in ths short run, rather than being determined by competitive entry. In that case, a firm's expected value from recruiting is endogenous, rather than being given by the vacancy cost k. All effects of the distributions on equilibrium contracts come through this expected value of recruiting, and these effects vanish in the long run when entry becomes competitive. Second, if there is exogenous separation between a worker-firm pair and the worker returns to unemployment after such exogenous separation, then the increase in unemployment benefits will affect optimal contracts by affecting the equation for $\dot{V}$.

[15]One can verify the statements here by introducing a continuous distribution of unemployment benefits into BC or Burdett and Mortensen (1998).

increase more quickly with tenure than before. As is clear from this explanation, the main cause for this difference is that the offer distribution plays a critical role in determining workers' quit rates under undirected search, but not so under directed search.

Now let me turn to the effects of an increase in λ_0, the probability with which an unemployed worker receives a job application opportunity. Again, the functions $w(.)$, $p(.)$, $q(.)$, $J(.)$, $F(.)$, and $D(.)$, are all unaffected, because the analysis in section 5 does not depend on λ_0. However, the function $V_u(.)$ does depend on λ_0. An increase in λ_0 increases $V_u(b)$ for any given b. Thus, v_0 increases, and so does v_1. The distributions of unemployed and employed workers change as well.

The increases in λ_0 has only a limited effect on the job-to-job transition rate and the wage path. To see this, let $\hat{v}_1$ be the new level of v_1 after the increase in λ_0. Because $w(.)$, $p(.)$ and $q(.)$ are unaffected, the optimal baseline contract after the increase in λ_0 is the part of the original baseline contract from $\hat{v}_1$ onward. Put differently, the new set of optimal contracts is identical to the subset of the original contracts whose starting values are equal to or greater than $\hat{v}_1$. Therefore, the job-to-job transition rate and the wage path of a worker to whom the contract provides $\hat{v}_1$ or more do not change. Again, these results contrast with those in undirected search models, where an increase in λ_0 increases the job-to-job transition rate and the steepness of the wage path.

The comparative statics above have obvious policy implications. If policymakers attempt to affect the labor market outcomes of employed workers, changing the aspects of the market for unemployed workers would be a wrong place to put the resource. Instead, the policy should directly target the aspects of the labor market relevant for employed workers, such as λ_1.

8. Conclusion

In this paper, I analyze the equilibrium in a labor market where firms offer wage-tenure contracts to direct the search of employed and unemployed workers. Each applicant observes all offers and there is no coordination among individuals. Because search is directed, workers' applications (as well as firms' recruiting decisions) must be optimal. This optimality requires the equilibrium to be formulated differently from the that in the large literature of undirected search. I provide such a formulation and show that the equilibrium exists. In the equilibrium, individuals explicitly tradeoff between an offer and the matching rate at that offer. This tradeoff yields a unique offer which is optimal for each worker to apply. Despite this uniqueness and directed search, the stationary equilibrium has a non-degenerate and continuous distribution of wages.

One cause of wage dispersion in the model is the feature that the optimal application increases with the value that a worker's current state yields. Even if all workers were initially identical, those who obtained jobs earlier will apply to higher wages than those who obtain jobs later. In the stationary equilibrium, a continuum of values are offered, each of which is tailored to workers who have a particular current value. The other cause of the wage distribution is the wage-tenure contract. With risk-averse workers and imperfect capital markets, it is optimal for a firm to offer a wage profile that increases smoothly with tenure. Such a contract provides partial insurance to the worker and backloads wages to increase retention of the worker. The positive wage-tenure relationship implies that workers who are employed under the same contract but at different times may earn different wages. It also implies that the quit rate falls with tenure.

While preserving the realistic predictions of undirected search models that wages increase, and quit rates fall, with tenure, the current model generates several novel implications. First, because applicants separate themselves according to their current values, wage mobility is endogenously limited by the workers' current wages. Second, the density function of the wage distribution of employed workers over wages is increasing at low wages and decreasing at high wages, even when all worker-firm pairs are equally productive. Finally, an increase in unemployment benefits has no effect on the set of equilibrium contracts or individual workers' job-to-job transition rates, although it affects wage distributions of workers.

This paper provides a tractable model of on-the-job search that will be useful for business cycle research. A striking feature of the equilibrium is that individuals' decisions and equilibrium contracts can all be characterized first without any reference to the distributions of workers and firms. This feature is a unique feature of directed search. With undirected search, instead, the distributions are state variables in every individual's decision problem. As the distributions evolve endogenously over business cycles, the large dimensionality of the state variables makes it intractable to solve for the dynamic equilibrium with on-the-job search, analytically or quantitatively. The above-mentioned feature of the current model eliminates such a difficulty. It is not difficult to see that the feature will continue to hold after aggregate and match-specific shocks are introduced into the model. Using this feature, Menzio and Shi (2007) examine the implications of on-the-job search on business cycles.

Appendix

A. Proof of Lemma 3.2

The result $F(\bar{V}) = \bar{V}$ is evident. Let $V < \bar{V}$ in the following proof. Temporarily denote $K(f, V) = p(f)(f - V)$. Because $p(.)$ is continuous and bounded, as stated in Lemma 3.1, $K(f, V)$ is continuous and bounded. Thus, the maximization problem in (3.1) has a solution. Because $p(\bar{V}) = 0$, I have $K(\bar{V}, V) = 0 = K(V, V)$. Since any interior value of f gives a positive value of $K(f, V)$, then the solution is interior. To show that the solution is unique, I show that $K(f, V)$ is strictly concave in f for all $f \in (V, \bar{V})$. To do so, let $\alpha \in (0, 1)$. Let f_1 and f_2 be two arbitrary interior values with $f_2 > f_1 > V$. Denote $f_\alpha = \alpha f_1 + (1 - \alpha) f_2$. Then,

$$
\begin{aligned}
K(f_\alpha, V) &= p(f_\alpha)\left[\alpha(f_1 - V) + (1 - \alpha)(f_2 - V)\right] \\
&\geq \left[\alpha p(f_1) + (1 - \alpha) p(f_2)\right]\left[\alpha(f_1 - V) + (1 - \alpha)(f_2 - V)\right] \\
&= \alpha K(f_1, V) + (1 - \alpha) K(f_2, V) + \alpha(1 - \alpha)[p(f_1) - p(f_2)][f_2 - f_1] \\
&> \alpha K(f_1, V) + (1 - \alpha) K(f_2, V).
\end{aligned}
$$

The two equalities come from rewriting, the first inequality from concavity of p, and the last inequality from the feature that $p(f)$ is strictly decreasing. Thus, $K(f, V)$ is strictly concave in f, which establishes part (i) of the Lemma.

For part (ii), uniqueness of the solution implies that $F(.)$ is continuous by the Theorem of the Maximum. To show that $D(.)$ is differentiable, let V_1 and V_2 be two arbitrary values with $V_1 < V_2 < \bar{V}$. Express $F_i = F(V_i)$ for $i = 1, 2$. Uniqueness of the solution implies $K(F_1, V_1) > K(F_2, V_1)$ and $K(F_2, V_2) > K(F_1, V_2)$. Thus,

$$D(V_2) - D(V_1) > K(F_1, V_2) - K(F_1, V_1) = -p(F_1)(V_2 - V_1);$$

$$D(V_2) - D(V_1) < K(F_2, V_2) - K(F_2, V_1) = -p(F_2)(V_2 - V_1).$$

Divide the two inequalities by $(V_2 - V_1)$ and take the limit $V_2 \to V_1$. Because $F(.)$ is continuous, the limit shows that $D(V)$ is differentiable at V_1 and that $D'(V_1) = -p(F_1)$. Since V_1 is arbitrary, this argument establishes part (ii).

For part (iii), again take two arbitrary values V_1 and V_2, with $V_1 < V_2 \leq \bar{V}$. Then, $p(F_j)(F_j - V_i) < p(F_i)(F_i - V_i)$ for $j \neq i$. I have:

$$
\begin{aligned}
0 &> [p(F_2)(F_2 - V_1) - p(F_1)(F_1 - V_1)] + [p(F_1)(F_1 - V_2) - p(F_2)(F_2 - V_2)] \\
&= p(F_2)(V_2 - V_1) + p(F_1)(V_1 - V_2) = [p(F_2) - p(F_1)](V_2 - V_1).
\end{aligned}
$$

This result implies $p(F_2) < p(F_1)$. Because $p(.)$ is strictly decreasing, $F(V_2) > F(V_1)$.

For part (iv), note that differentiability of p implies that $F(V)$ is given by the first-order condition of the maximization problem, which leads to (3.2). Also, because p is concave and decreasing, the following inequalities hold for all V_1 and V_2 with $V_2 > V_1$:

$$p(F_1) \geq p(F_2) - p'(F_2)\left(F_2 - F_1\right), \; p(F_2) \geq p\left(F_1\right) + p'(F_2)\left(F_2 - F_1\right)$$

where $F_i = F(V_i)$, $i = 1, 2$. Substituting the first inequality into (3.2) yields:

$$V_2 - V_1 \geq 2\left(F_2 - F_1\right) + p\left(F_2\right)\frac{p'\left(F_1\right) - p'\left(F_2\right)}{p'\left(F_1\right)p'\left(F_2\right)} \geq 2\left(F_2 - F_1\right).$$

This implies $\left(F_2 - F_1\right)/\left(V_2 - V_1\right) \leq 1/2$ for all $V_2 \neq V_1$, and so F is Lipschitz. If, in addition, p is twice differentiable, then differentiating the first-order condition generates the derivative $F'(V)$ and the above Lipschitz property yields $F'(V) \leq 1/2$. In this case, $D''(V) = -p'(F(V))F'(V)$. QED

B. Proofs of Lemmas 3.3 and 5.1

I prove Lemma 3.3 first. By Lemmas 3.1 and 3.2, $p'(F(V)) < 0$ and $F'(V) > 0$ for all $V < \bar{V}$. Because $J(V) > 0$ for all V, as shown later, then (3.8) implies $\dot{w}(t) > 0$ for all $V < \bar{V}$. Because $\bar{V}$ is the highest value offered, then $p(F(\bar{V})) = 0$ and $\dot{V} = 0$ at $V = \bar{V}$. Then $D(\bar{V}) = 0$, and (3.3) implies $\bar{V} = u(\bar{w})/\sigma$. Similarly, because $\dot{J}(\bar{V}) = 0$, (3.5) implies $J(\bar{V}) = (y - \bar{w})/\sigma$. Because recruiting at $\bar{w}$ should yield zero net profit, $q(\bar{V})J(\bar{V}) = k$; that is, $\bar{w} = y - \sigma k/q(\bar{V})$. If $q(\bar{V}) = \bar{q}$, then the stated expressions for $\bar{w}$ and $J(\bar{V})$ follow. Since $\bar{q} < \infty$ by Assumption 2, then $\bar{w} < y$ and $J(\bar{V}) > 0$.

To show $q(\bar{V}) = \bar{q}$, suppose that $q(\bar{V}) = \bar{q} - \delta$ to the contrary, where $\delta > 0$. Because $q(\bar{V})J(\bar{V}) = k > 0$ and $J(\bar{V}) = (y - \bar{w})/\sigma$, then $\bar{w} = y - \sigma k/(\bar{q} - \delta)$. Consider a firm that deviates from $\bar{w}$ to $\bar{w} + \varepsilon$, where $\varepsilon > 0$, which generates a value to a worker as $\hat{V} = u(\bar{w} + \varepsilon)/\sigma$. Because the firm is the only one that offers a wage higher than $\bar{w}$, the workers who are employed at $\bar{w}$ will all apply to this firm, which yields $q(\hat{V}) = \bar{q}$. The firm's expected value of recruiting is $q(\hat{V})J(\hat{V}) = (y - \bar{w} - \varepsilon)\bar{q}/\sigma$, which exceeds k for sufficiently small $\varepsilon > 0$. This result contradicts the fact that $\bar{V}$ is an equilibrium value. Thus, $q(\bar{V}) = \bar{q}$. This completes the proof of Lemma 3.3.

Now, turn to Lemma 5.1. Let $w(V)$ be an arbitrary function in Ω. It is easy to verify that $J_w(V)$ defined by (5.1) is strictly positive, bounded, strictly decreasing and continuously differentiable, with $J'(V) = -1/u'(w(V)) < 0$. Because $w(V)$ is increasing, then $J'(V)$ is decreasing and $J(V)$ is (weakly) concave. Moreover, $J_w(\bar{V}) = k/\bar{q}$. Similarly, $p_w(V)$ defined by (5.2) is bounded and continuous for all V (including $V = \bar{V}$), with $p_w(\bar{V}) = P(\bar{q}) = 0$. For all $V < \bar{V}$, $p_w(V)$ is differentiable and strictly decreasing because

$$p_w'(V) = \left(P'\frac{k}{J_w^2}\right)\frac{1}{u'(w(V))} < 0,$$

where the argument of P' is $k/J_w(V)$ and $P' < 0$ under Assumption 2. Moreover, for any given value V,

$$\frac{d}{dJ_w}\left(P'\frac{k}{J_w^2}\right) = \frac{k}{J_w^3}\left(-\frac{k}{J_w}P'' - 2P'\right) \geq 0$$

where the inequality follows from part (iii) of Assumption 2. Because $J_w(V)$ is decreasing, the function $P'k/J_w(V)$ is decreasing. Because $1/u'(w(V))$ is increasing in V and $P' < 0$, then $p_w'(V)$ is decreasing. That is, $p_w(V)$ is (weakly) concave. QED

C. Proof of Theorem 5.2

The sets Ω and Ω' are defined prior to Lemma 5.1 and the mapping Γ is defined by $w_1(V) = (\Gamma w)(V)$, where w_1 is the solution to (5.3). It can be verified that Ω is a closed and convex set. Lemmas C.1 and C.2 below state that $\Gamma : \Omega \to \Omega'$ is a continuous mapping in the supnorm. Under the assumption that the image of Γ is compact, the Schauder fixed point theorem implies that Γ has a fixed point in Ω, denoted as w^*. Because $w^*(V) = (\Gamma w^*)(V) \in \Omega'$, then $w^*(V)$ is strictly increasing for all $V < \bar{V}$. This implies that $J_{w^*}(V)$ and $p_{w^*}(V)$ are strictly concave, in addition to the properties stated in Lemma 5.1.

Lemma C.1. $\Gamma : \Omega \to \Omega' \subset \Omega$.

Proof. Temporarily denote the left-hand side of (5.3) as $L(w_1)$ and the right-hand side as $R(V)$. Recall that $\bar{w} < y$. Because $L(w)$ is continuous and strictly decreasing for all $w < y$, it is invertible for all $w \in [\underline{w}, \bar{w}]$. Then, $w_1(V) = L^{-1}(R(V))$. Pick an arbitrary $w \in \Omega$. I show that $w_1 \in \Omega'$. This is done in the following steps.

First, $w_1(V)$ is continuous because $J_w(.)$, $p_w(.)$ and $F_w(.)$ are all continuous.

Second, $w_1(V)$ is strictly increasing for all $V < \bar{V}$, i.e., $R(V)$ is strictly decreasing. Pick arbitrary values V_1 and V_2, with $\underline{V} \leq V_1 < V_2 < \bar{V}$, we show that the following (stronger) property holds:

$$0 < J_w(V_2)\, S \leq R(V_1) - R(V_2) \leq J_w(V_1)\, S, \tag{C.1}$$

where

$$S = u'(w(V_2))\,[\sigma + \lambda_1 p_w(F_2)] - u'(w(V_1))\,[\sigma + \lambda_1 p_w(F_1)] > 0.$$

Note that $S > 0$, indeed, because $w(V)$ is increasing, $p_w(F)$ is strictly decreasing and $F_w(V)$ is strictly increasing. To establish (C.1), note that $J_w(V)$ is decreasing and concave with derivative $J'_w(V) = -1/u'(w(V))$. Then,

$$\frac{V_2 - V_1}{u'(w(V_1))} \leq J_w(V_1) - J_w(V_2) \leq \frac{V_2 - V_1}{u'(w(V_2))},$$

Similarly, because the function $[\sigma V - \lambda_1 D_w(V)]$ is increasing and concave with derivative $-[\sigma + \lambda_1 p_w(F)]$, I have:

$$-[\sigma + \lambda_1 p_w(F_1)] \leq \frac{[\sigma V - \lambda_1 D_w(V_1)] - [\sigma V - \lambda_1 D_w(V_2)]}{V_2 - V_1} \leq -[\sigma + \lambda_1 p_w(F_2)].$$

Using the first part of the above two results to substitute $J_w(V_1)$ and $[\sigma V - \lambda_1 D_w(V_1)]$ in $R(V_1)$, I get $R(V_1) - R(V_2) \geq J_w(V_2)\, S$. Using the second part of the above two results to substitute $J_w(V_2)$ in $R(V_1)$, I get the second part of $R(V_1) - R(V_2) \leq J_w(V_1)\, S$.

Third, $w_1(V) \in [\underline{w}, \bar{w}]$ for all V, with $w_1(\bar{V}) = \bar{w}$. Examine $w_1(\bar{V})$. Because $w(\bar{V}) = \bar{w}$, then (5.3) implies:

$$L(w_1(\bar{V})) = R(\bar{V}) = u'(\bar{w})(y - \bar{w}) + u(\bar{w}) = L(\bar{w}).$$

Because $L(w)$ is strictly decreasing, the above equation implies $w_1(\bar{V}) = \bar{w}$. Since $w_1(V)$ is strictly increasing for $V < \bar{V}$, then $w_1(V) < \bar{w}$ for all $V < \bar{V}$.

Finally, I show $w_1(V) \geq \underline{w}$. Since $L'(w) < 0$, $w_1(V) \geq \underline{w}$ if and only if $L(\underline{w}) \geq R(V)$. A sufficient condition is $L(\underline{w}) \geq R(\underline{V})$, because $R(V)$ is a decreasing function. Note that the following holds:

$$\begin{aligned} R(\underline{V}) &= u'(w(\underline{V}))\left[\sigma + \lambda_1 p_w(F_w(\underline{V}))\right] J_w(\underline{V}) + \sigma \underline{V} - \lambda_1 D_w(\underline{V}) \\ &< u'(\underline{w})\left[\sigma + \lambda_1 p_w(F_w(\underline{V}))\right] J_w(\underline{V}) + u(\underline{b}) \\ &\leq u'(\underline{w})\left[\sigma + \lambda_1 p_w(\underline{V})\right] J_w(\underline{V}) + u(\underline{b}) \\ &\leq u'(\underline{w})\left[\sigma + \lambda_1 p_{\bar{w}}(\underline{V})\right] J_{\bar{w}}(\underline{V}) + u(\underline{b}) \end{aligned}$$

The first inequality follows from the facts that $w(\underline{V}) \geq \underline{w}$, $\underline{V} = u(\underline{b})/\sigma$ and $D_w(\underline{V}) > 0$. The second inequality follows from the facts that $F_w(\underline{V}) \geq \underline{V}$ and that $p_w(.)$ is decreasing. To obtain the third inequality, note that $J_w(V) \leq J_{\bar{w}}(V)$ and $p_w(V) \leq p_{\bar{w}}(V)$ for all V. Therefore, a sufficient condition for $w_1(V) \geq \underline{w}$ is:

$$L(\underline{w}) \geq u'(\underline{w})\left[\sigma + \lambda_1 p_{\bar{w}}(\underline{V})\right] J_{\bar{w}}(\underline{V}) + u(\underline{b})$$

This condition can be re-arranged as (5.7), which is assumed to hold. This completes the proof of Lemma C.1.

Lemma C.2. Γ *is continuous in the supnorm.*

Proof. To show that the mapping Γ is continuous in the supnorm, I show that the following holds for all $w_a, w_b \in \Omega$ and all V:

$$|(\Gamma w_a)(V) - (\Gamma w_b)(V)| \leq A \|w_a - w_b\|, \tag{C.2}$$

where the norm is the supnorm and $A > 0$ is a finite constant. Once this is done, then

$$\|\Gamma w_a - \Gamma w_b\| = \sup |(\Gamma w_a)(V) - (\Gamma w_b)(V)| \leq A \|w_a - w_b\|,$$

which implies that Γ is continuous in the supnorm.

To show (C.2), take arbitrarily $w_a, w_b \in \Omega$ and $V \in [\underline{V}, \bar{V}]$. Without loss of generality, assume $w_a(V) \geq w_b(V)$ for the given value V. Shorten the subscript w_i on J, p, F, and D to i, where $i = a, b$. Also, denote the right-hand side of (5.3) with $w = w_i(V)$ as $R_i(V)$. Because $w \geq w_L > 0$, Assumption 1 implies that there are positive and finite constants ω_1 and ω_2 such that $\omega_1 \leq |u''(w)| \leq \omega_2$ for all $w \in [\underline{w}, \bar{w}]$. Then

$$|L'(w)| = (y - w)\,|u''| \geq (y - \bar{w})\,\omega_1 \equiv A_1.$$

Note that A_1 is bounded above 0. Since $L(w)$ is decreasing, then

$$|R_a(V) - R_b(V)| = |L(\Gamma w_a(V)) - L(\Gamma w_b(V))| \geq A_1 |\Gamma w_a(V) - \Gamma w_b(V)|.$$

I show that $|R_a(V) - R_b(V)| \leq A_6 \|w_a - w_b\|$ for some positive and finite A_6. Then, (C.2) holds after defining $A = A_6/A_1$. To establish the desired inequality for R, suppress the given V. I have:

$$\begin{aligned}
|R_a - R_b| &= |\{[u'(w_a) - u'(w_b)]\, J_a + u'(w_b)(J_a - J_b)\}\, [\sigma + \lambda_1 p_a(F_a)] \\
&\quad + \lambda_1 u'(w_b) J_b\, [p_a(F_a) - p_b(F_b)] - \lambda_1\, [D_a - D_b]| \\
&\leq [|u'(w_a) - u'(w_b)|\, J_a + u'(w_b)\, |J_a - J_b|]\, [\sigma + \lambda_1 p_a(F_a)] \\
&\quad + \lambda_1 u'(w_b) J_b\, |p_a(F_a) - p_b(F_b)| + \lambda_1\, |D_a - D_b|
\end{aligned}$$

I find the bound on each of the absolute values in the above expression.

Because $u'' < 0$, then

$$|u'(w_a) - u'(w_b)| \leq |w_a - w_b| \max\{|u''(w_a)|\,, |u''(w_b)|\} \leq \omega_2\, \|w_a - w_b\| \tag{C.3}$$

By the definition of J_w,

$$\begin{aligned}
|J_a - J_b| &= \left| \int_{\underline{V}}^{\bar{V}} \tfrac{u'(w_a(z)) - u'(w_b(z))}{u'(w_a(z)) u'(w_b(z))} dz \right| \\
&\leq \tfrac{1}{[u'(\bar{w})]^2} \int_{\underline{V}}^{\bar{V}} |u'(w_a(z)) - u'(w_b(z))|\, dz \\
&\leq \tfrac{\omega_2}{[u'(\bar{w})]^2} \int_{\underline{V}}^{\bar{V}} |w_a(z) - w_b(z)|\, dz \leq \tfrac{\omega_2 (\bar{V} - \underline{V})}{[u'(\bar{w})]^2}\, \|w_a - w_b\|
\end{aligned} \tag{C.4}$$

The coefficient of $\|w_a - w_b\|$ is bounded because $u'(\bar{w}) > 0$ and $\omega_2 < \infty$.

To develop bounds on $|p_a(F_a) - p_b(F_b)|$ and $|D_a - D_b|$, let $\varepsilon = \|w_a - w_b\| > 0$ with loss of generality. (If $\|w_a - w_b\| = 0$, then $w_a = w_b$ for all V, in which case $|p_a(F_a) - p_b(F_b)| = |D_a - D_b| = \|w_a - w_b\|$; these provide the required bounds.) I examine two cases separately: the case where V is close to $\bar{V}$ and the case where V is away from $\bar{V}$. The separation is necessary because $P'(q)$ and $P''(q)$ might be unbounded at $q = \bar{q}$ (i.e., at $V = \bar{V}$).

Consider first the case where V is close to $\bar{V}$. In this case, $F_a(V)$ and $F_b(V)$ are close to $\bar{V}$. Because $p_w(V)$ is continuous for all V, including $V = \bar{V}$, and because $F(V)$ is continuous, then for given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\bar{V} - V < \delta \Longrightarrow \left| p_i(F_i) - p_i(\bar{V}) \right| < \varepsilon/2, \quad \text{for } i \in \{a, b\}$$

Because $p_i(\bar{V}) = 0$, the following holds for $V > \bar{V} - \delta$:

$$|p_a(F_a) - p_b(F_b)| \leq |p_a(F_a)| + |p_b(F_b)| < \varepsilon = \|w_a - w_b\| \tag{C.5}$$

$$|D_a - D_b| \leq |p_a(F_a)|\, (F_a - V) + |p_b(F_b)|\, (F_b - V) < (\bar{V} - \underline{V})\, \|w_a - w_b\| \tag{C.6}$$

For the last inequality, I used the facts that $|p_i(F_i)| < \varepsilon/2$ and that $F_i - V_i \leq \bar{V} - \underline{V}$. (C.5) and (C.6) provide the required bounds when $V > \bar{V} - \delta$.

Now consider the case where $V \leq \bar{V} - \delta$, where $\delta > 0$. In this case, $q < \bar{q}$, and hence Assumption 2 implies that $|P'(q)|$ and $|P''(q)|$ are bounded for $q \in [\underline{q}, \bar{q})$. Because $p(V) = P\left(\frac{k}{J(V)}\right)$, then

$$\left| \frac{dP(k/J)}{dJ} \right| = \left(-\frac{k}{J^2} \right) P'\left(\frac{k}{J} \right)$$

$$\left|\frac{d^2P(k/J)}{dJ^2}\right| = \left(\frac{k}{J_w^3}\right)\left(-\frac{k}{J_w}P'' - 2P'\right)$$

These absolute values are bounded above in the current case. Let A_2 and A_3 be the upper bounds. Define

$$A_4 = A_2\frac{\omega_2(\bar{V} - \underline{V})}{[u'(\bar{w})]^2} < \infty$$

For any $x \in [\underline{V}, \bar{V} - \delta]$,

$$|p_a(x) - p_b(x)| \leq A_2 \, |J_a(x) - J_b(x)| \leq A_4 \, \|w_a - w_b\|$$

$$\left|\frac{dP_a}{dJ_a} - \frac{dP_b}{dJ_b}\right| \leq A_3 \, |J_a - J_b|$$

These results lead to the following result:

$$\begin{aligned}
|p_a'(x) - p_b'(x)| &\leq \left|\frac{dP_a/dJ_a}{u'(w_a)} - \frac{dP_b/dJ_a}{u'(w_b)}\right| \\
&\leq \left|\frac{dP_a}{dJ_a}\right|\left|\frac{1}{u'(w_a)} - \frac{1}{u'(w_b)}\right| + \frac{1}{u'(w_b)}\left|\frac{dP_a}{dJ_a} - \frac{dP_b}{dJ_b}\right| \\
&\leq \frac{A_2}{[u'(\bar{w})]^2} \, |u'(w_a) - u'(w_b)| + \frac{A_3}{u'(\bar{w})} \, |J_a - J_b| \\
&\leq \frac{A_4}{\bar{V} - \underline{V}} \, \|w_a - w_b\| + \frac{A_4 A_3/A_2}{u'(\bar{w})} \, \|w_a - w_b\|
\end{aligned}$$

Suppose first that $F_a \geq F_b$. If $p_a(F_a) \geq p_b(F_b)$, then

$$0 \leq p_a(F_a) - p_b(F_b) \leq p_a(F_a) - p_b(F_a) \leq A_4 \, \|w_a - w_b\|$$

The second inequality comes from the fact that p is decreasing and the last inequality from the bound on $|p_a - p_b|$ just derived. If $p_a(F_a) < p_b(F_b)$, then

$$\begin{aligned}
0 &< p_b(F_b) - p_a(F_a) = -p_b'(F_b)(F_b - V) + p_a'(F_a)(F_a - V) \\
&\leq (F_a - V)\,[p_a'(F_a) - p_b'(F_b)] \leq (\bar{V} - \underline{V})\,[p_a'(F_b) - p_b'(F_b)] \\
&\leq \left[1 + \frac{A_3(\bar{V} - \underline{V})}{A_2 u'(\bar{w})}\right] A_4 \, \|w_a - w_b\|
\end{aligned}$$

The equality follows from the first-order condition for F, the second inequality from the supposition $F_a \geq F_b$, the third inequality from the facts that p' is a decreasing function and that $F_a - V \leq \bar{V} - \underline{V}$, and the last inequality from the bound on $|p_a' - p_b'|$. Thus, if $F_a \geq F_b$, then

$$|p_a(F_a) - p_b(F_b)| \leq \left[1 + \frac{A_3(\bar{V} - \underline{V})}{A_2 u'(\bar{w})}\right] A_4 \, \|w_a - w_b\| \tag{C.7}$$

Suppose now that $F_a < F_b$. By switching the roles of F_a and F_b, it can be shown that (C.7) continues to hold. Thus, (C.7) holds for arbitrary $F_a(V)$ and $F_b(V)$ with $V \leq \bar{V} - \delta$.

Now let us examine $|D_a - D_b|$ for the case $V \leq \bar{V} - \delta$. If $D_a \geq D_b$, then

$$\begin{aligned}
0 &\leq D_a - D_b = p_a(F_a)(F_a - V) - p_b(F_b)(F_b - V) \\
&\leq p_a(F_a)(F_a - V) - p_b(F_a)(F_a - V) \\
&= (F_a - V)\,[p_a(F_a) - p_b(F_a)] \leq (\bar{V} - \underline{V})A_4 \, \|w_a - w_b\|
\end{aligned}$$

The first equality comes from the definition of $D(V)$, the second inequality from the fact that $p_b(f)(f-V)$ is maximized at $f=F_b$, the last inequality from the bound on $|p_a-p_b|$ derived above and the fact $F_a - V \leq \bar{V} - \underline{V}$. The same result holds if $D_a < D_b$. Thus,

$$|D_a - D_b| \leq (\bar{V} - \underline{V}) A_4 \|w_a - w_b\| \tag{C.8}$$

Defining $A_5 = \max\{A_4, 1\}$ and replace A_4 in (C.7) and (C.8) with A_5. The resulting bounds on $|p_a - p_b|$ and $|D_a - D_b|$ apply for both the case $V > \bar{V} - \delta$ and $V \leq \bar{V} - \delta$. Substituting these bounds, (C.3) and (C.4), I have:

$$\begin{aligned} |R_a - R_b| \;\; & \leq \Big\{ \Big[\omega_2 J_a + u'(w_b)\tfrac{A_4}{A_2}\Big] [\sigma + \lambda_1 p_a(F_a)] \\ & \quad + \lambda_1 A_5 \Big[u'(w_b)\Big(1 + \tfrac{A_3(\bar{V}-\underline{V})}{A_2 u'(\bar{w})}\Big) J_b + \lambda_1(\bar{V} - \underline{V})\Big]\Big\} \|w_a - w_b\| \end{aligned}$$

Let A_6 be the maximum value of the coefficient of $\|w_a - w_b\|$ in the above expression, taken over $V \in [\underline{V}, \bar{V}]$. Then, A_6 is bounded above. Setting $A = A_6/A_1$ establishes the inequality (C.2), which shows that Γ is continuous in the supnorm. This completes the proof of Lemma C.2, and hence of Theorem 5.2. QED

D. Proofs of Corollary 5.3 and Lemma 6.1

To prove Corollary 5.3, suppose that $|\dot{w}(t)| < \infty$ for all t. That is, $\dot{w}(V(t))$ is finite. If $\dot{V} \neq 0$, then $w'(V) = \dot{w}/\dot{V}$ exists and is finite. If $\dot{V} = 0$ at V_1, then $\sigma V_1 - u(w(V_1)) - \lambda_1 D(V_1) = 0$. Differentiating this equation with respect to V_1 yields:

$$w'(V_1) = \frac{\sigma + \lambda_1 p(F(V_1))}{u'(w(V_1))} \in (0, \infty). \tag{D.1}$$

That is, $w(V)$ is differentiable at V_1. This argument applies to $\bar{V}$, because $\dot{V} = 0$ at $V = \bar{V}$. Thus, $w'(V_1)$ exists and is finite for all V. From (5.1), (5.2) and Lemma 3.2, one can then verify that $J''(V)$, $p''(V)$ and $F'(V)$ all exist and are finite for all $V < \bar{V}$.

I still need to show that $w'(V) > 0$, $\dot{V} > 0$ and $\dot{J}(V) < 0$ in the case $V < \bar{V}$. In this case, $F(V) < \bar{V}$. Lemma 3.2 implies $dp(F(V))/dV < 0$. The right-hand side of (3.8) is positive and finite, which implies $\dot{w}(V) > 0$. Thus, $w'(V)\dot{V} \in (0, \infty)$ for all $V < \bar{V}$. Because $w(V)$ is strictly increasing for all $V < \bar{V}$ and $\dot{V}$ is bounded (see (3.3)), then $w'(V) \in (0, \infty)$ and $\dot{V} \in (0, \infty)$ for all $V < \bar{V}$. Finally, $\dot{J}(V) = J'(V)\dot{V} \in (0, \infty)$ for all $V < \bar{V}$. This completes the proof of Corollary 5.3.

For Lemma 6.1, consider the group of unemployed workers whose values are greater than V, where $V \in [v_0, \bar{V}]$. Equating the flows into this group to the flows out of this group in a small interval of time dt, I obtain:

$$\begin{aligned} & (\sigma dt) \{1 - (1-n) [1 - G_u(V)]\} [1 - \Phi(V)] \\ = \;\; & (\sigma dt)(1-n)[1 - G_u(V)]\Phi(V) + \lambda_0 (1-n) \textstyle\int_V^{\bar{V}} [p(F(z))dt] dG_u(z). \end{aligned}$$

The left-hand side gives the flow into the group, which is generated by newborns who replace workers who were not in the group and who just died. The measure of agents who were not in the described group is $\{1-(1-n)[1-G_u(V)]\}$. When such an agent dies and is replaced by a new agent, the new agent belongs to the described group if the agent draws a value of leisure higher than V, which occurs with probability $1-\Phi(V)$. Note that if agents who just died were in the described group and are replaced by new agents who draw values above V, such newborns do not change the measure of the group. The right-hand side of the above equation gives the flows out of the group. The first term is caused by death in the group whose replacements draw values less than or equal to V. The second term is the flow of agents who just exited the group as the result of becoming employed (at higher values).

Dividing the two sides of the equation by dt and re-arranging, I obtain:

$$\sigma[n-\Phi(V)+(1-n)G_u(V)]=\lambda_0(1-n)\int_V^{\bar{V}} p(F(z))dG_u(z). \tag{D.2}$$

From this equation one can show that $G_u(V)$ is continuous and differentiable for all $V\in[v_0,\bar{V}]$. Differentiating with respect to V, I get:

$$g_u(V)=\frac{\sigma\phi(V)}{(1-n)[\sigma+\lambda_0 p(F(V))]}. \tag{D.3}$$

Integrating over V yields (6.3). Because $G_u(\bar{V})=1$, the fraction of employed workers satisfies (6.4). QED

E. Proofs of Theorem 6.2 and Corollary 6.3

To prove Theorem 6.2, I establish continuity of G_e first. Use (D.2) to substitute for the last term in (6.5). With Δ being defined prior to (6.6), I get:

$$\begin{aligned} & \lim_{dt\downarrow 0}\tfrac{G_e(V)-G_e(V-\dot{V}dt)}{dt} \\ = \;& \sigma\Delta(F^{-1}(V))-\sigma G_e(V)-\lambda_1\textstyle\int_{\max\{v_1,F^{-1}(V)\}}^{V} p(F(z))dG_e(z) \end{aligned} \tag{E.1}$$

Suppose, contrary to the theorem, that G_e has a mass point at a value $V\in[v_1,\bar{V}]$. Then

$$\lim_{dt\downarrow 0}\frac{G_e(V)-G_e(V-\dot{V}dt)}{dt}=\infty.$$

This violates (E.1), because the right-hand side of (E.1) is bounded. Thus, G_e does not have a mass point, and so it is a continuous function.

The density function, g_e, is also continuous. To see this, denote

$$g_e(V_-)=\lim_{dt\downarrow 0}\frac{G_e(V)-G_e(V-dt)}{dt}$$

The left-hand side of (E.1) is equal to $g_e(V_-)\dot{V}$. The right-hand side is continuous in V, because G_e, F^{-1} and $p(F(.))$ are continuous. Thus, $g_e(V_-)\dot{V}$ must be continuous. Because $\dot{V}$ is also continuous, g_e must be continuous. Thus, I can rewrite (E.1) as (6.7).

Continuity of g_e implies that G_e is continuously differentiable. Since F^{-1} and $p(F(.))$ are continuously differentiable, differentiability of G_e implies that the right-hand side of (6.7) is continuously differentiable for all $V \neq v_2$. Therefore, g_e is continuously differentiable for all $V \neq v_2$.

Now I derive (6.8) and (6.9). For $V \in [v_1, v_2)$, $F^{-1}(V) < v_1$, and so (6.7) becomes:

$$g_{e1}(V)\dot{V} = \sigma\Delta(F^{-1}(V)) - \sigma G_{e1}(V) - \lambda_1 \int_{v_1}^{V} p(F(z))dG_{e1}(z) \tag{E.2}$$

Setting $V = v_1$ in (E.2) leads to $g_{e1}(v_1) = 0$. Differentiate (E.2) and divide the result by $\gamma(T(V), 0)$, where γ is defined by (3.6) and T by (3.11). I have:

$$\frac{g'_{e1}(V)\dot{V} + a(V)g_{e1}(V)}{\gamma(T(V), 0)} = \frac{b(V)}{\gamma(T(V), 0)} \tag{E.3}$$

where

$$a(V) = \sigma + \lambda_1 p(F(V)) + \frac{d\dot{V}}{dV}, \quad b(V) = \sigma\frac{d}{dV}\Delta(F^{-1}(V))$$

The definition of $T(V)$ implies $T'(V) = 1/\dot{V}$. Then, it can be verified that the left-hand side of (E.3) is equal to the derivative of the function, $g_{e1}(V)\dot{V}/\gamma(T(V), 0)$, with respect to V. Integrate (E.3) from v_1 to V. Using the fact $\gamma(T(V), 0)/\gamma(T(z), 0) = \gamma(T(V), T(z))$ to rewrite the result, I have (6.8). Since g_e is continuous, taking the limit $V \uparrow v_2$ in (6.8) gives $g_e(v_2)$.

Now examine the case $V \in [v_j, v_{j+1})$, where $j \geq 2$. In this case, $F^{-1}(V) \geq v_1$, and so (6.7) becomes

$$g_{ej}(V)\dot{V} = \sigma\Delta(F^{-1}(V)) - \sigma G_{en}(V) - \lambda_1 \int_{F^{-1}(V)}^{V} p(F(z))dG_e(z) \tag{E.4}$$

I do not add the subscript j to G_e on the right-hand side of the equation because, if $v_j < V < v_{j+1}$, some applicants to values above V come from the interval $[v_j, V)$ while others come from the interval $[F^{-1}(V), v_j)$. Differentiating (6.7) yields:

$$g'_{ej}(V)\dot{V} + a(V)g_{ej}(V) = b(V) + \lambda_1 p(V)g_{e(j-1)}(F^{-1}(V))\frac{dF^{-1}(V)}{dV} \tag{E.5}$$

where $a(V)$ and $b(V)$ are defined as before. Using the same procedure as the one used to solve for g_{e1} above, I obtain:

$$g_{ej}(V) = \frac{1}{\dot{V}}\int_{v_1}^{V} \gamma(T(V), T(z))\left\{\sigma\delta(F^{-1}(z)) + \lambda_1 p(z)g_{e(j-1)}(F^{-1}(z))\right\} dF^{-1}(z)$$

To obtain (6.9), set $V = v_j$ in the above solution:

$$g_{ej}(v_j) = \frac{1}{\dot{v}_j}\int_{v_1}^{v_j} \gamma(T(v_j), T(z))\left\{\sigma\delta(F^{-1}(z)) + \lambda_1 p(z)g_{e(j-1)}(F^{-1}(z))\right\} dF^{-1}(z)$$

Note that

$$\gamma\left(T(V), T(v_j)\right) \gamma(T(v_j), T(z)) = \gamma(T(V), T(z)).$$

Using this fact and the above formulas for $g_{ej}(V)$ and $g_{ej}(v_j)$, one can compute the left-hand side of (6.9) and show that it is equal to the right-hand side. Because g_e is continuous, then $g_{ej}(v_j) = \lim_{V \to v_j} g_{e(j-1)}(V)$, all j. This completes the proof of Theorem 6.2.

Now, turn to Corollary 6.3. I have shown $g_e(v_1) = 0$ in the above proof. Because $0 < F'(V) \leq 1/2$ for all $V < \bar{V}$, then $dF^{-1}(V)/dV > 0$ for all $V < \bar{V}$. Also, $\dot{V} > 0$ and $\delta(V) > 0$ for all $V < \bar{V}$. These features imply that $b(V) > 0$ for all $V < \bar{V}$. Substituting this result and $g_e(v_1) = 0$ into (E.5) yields $g_e'(v_1) > 0$.

Supposing $F'(\bar{V}) > 0$, I now show that $g_e(\bar{V}) = 0$. The supposition $F'(\bar{V}) > 0$ implies that $dF^{-1}(V)/dV$ is bounded. Because $p(\bar{V}) = 0$, then regardless of whether $\bar{V} = v_2$, the following holds (see (E.3) and (E.5)):

$$g_e'(\bar{V})\dot{V}|_{V=\bar{V}} + a(\bar{V})g_e(\bar{V}) = b(\bar{V})$$

The first term is zero because $\dot{V} = 0$ at $V = \bar{V}$. Since (D.1) holds for $V_1 = \bar{V}$, then

$$\left.\frac{d\dot{V}}{dV}\right|_{V=\bar{V}} = \sigma + \lambda_1 p(F(\bar{V})) - u'(w(\bar{V}))w'(\bar{V}) = 0.$$

This implies $a(\bar{V}) = \sigma$. Because $p(F(\bar{V})) = 0$, then $\delta(\bar{V}) = 0$. In addition, $dF^{-1}(V)/dV$ is finite at $V = \bar{V}$. Thus, $b(\bar{V}) = 0$. The above form of (E.5) at $V = \bar{V}$ becomes $0 = -\sigma g_e(\bar{V})$, i.e., $g_e(\bar{V}) = 0$.

The feature $g_e'(v_1) > 0$ implies that $g_e(v_1 + \varepsilon) > 0$, where $\varepsilon > 0$ is sufficiently small. Because $g_e(V)$ is continuous and $g_e(\bar{V}) = 0$, then $g_e(V)$ must be decreasing when V is close to $\bar{V}$. That is, there exists $\hat{V} \in (v_1, \bar{V})$ such that $g_e'(V) < 0$ for $V \in [\hat{V}, \bar{V})$. QED

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